Amortized Analysis

Thomas Schwarz, SJ

Amortized Analysis

- Determine the average time of a series of operations
  - Allows to optimize average performance
  - Example: Linear Hashing with fixed load factor α.
    - Cost of bucket split is divided over $\frac{1}{\alpha}$ inserts

Amortized Analysis

- Aggregate Analysis
  - Determine upper bound $T(n)$ on a sequence of n operations
  - Average cost is then $T(n)/n$
- Accounting method:
  - Most operations gets overcharged
  - Accumulated overcharges are used to pay for later operations
- Potential method:
  - Model the “credit” as a potential

Aggregate Analysis

- Stacks have
  - push
  - pop
  - empty
- Add a multipop(k) method that pops *k* elements (or empties the stack if there are less than *k* elements on the stack)
  - standard operations are $O(1)$
  - multipop is worst case $O(n)$

Aggregate Analysis

- Given a sequence of m stack operations on an initially empty stack
  - Naïve calculation:
    - At most m elements on the stack at one time
    - Therefore, worst case cost is $\sim m \times m = O(m^2)$
  - Better analysis:
    - Combined number of steps of pops and multi-pops is smaller or equal to the number of steps of pushes
    - Therefore: combined number of steps is at most m
    - Therefore: amortized costs $O(1)$

Aggregate Analysis

- Counters implemented as binary array
  - Interested in calculating the bit-flips

0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

  - Single increment can flip all the bits
  - Array with k binary registers has $2^k - 1$ increments
    - Does this mean that number of bit-flips for k increments is $\Theta(k)$ per increment?

Aggregate Analysis

- Write down binary numbers in order
- Observe
  - Least Significant Bit (LSB) flips every time
  - Second LSB flips every other time
  - Third LSB flips every 2^2 times
  - …

```
000000
000001
000010
000011
000100
000101
000110
000111
001000
001001
001010
001011
001100
001101
001110
```

Aggregate Analysis

- Assume *n* increment operations with an arbitrary starting counter value
- Then number of bit-flips is less than
  - $\sum_{i=0}^{k-1} \lfloor \frac{n}{2^i} \rfloor$
    - $< \sum_{i=0}^{k-1} \frac{n}{2^i}$
    - $< n \sum_{i=0}^{\infty} \frac{1}{2^i}$
    - $= n \times 0.11111\ldots_2 = 2n$

Aggregate Analysis

- Aggregate cost of *n* increment operations is therefore
  - $< 2n$
  - $= \Theta(n)$
- Average cost per increment is 2

Accounting Method

- Stack costs:
  - Push: 1
  - Pop: 1
  - Multipop(k): $\min(k, s)$ where s is the number of elements in the stack
- Charges:
  - Push: 2
  - Pop: 0
  - Multipop(k): 0

Accounting Method

- Show that charges will pay for all operations.
  - push pays for itself and for removing the element
  - since we cannot remove elements that have not been pushed, charged amount is always sufficient

Accounting Method

- Counter:
  - Charge 2 for every bit set to one
  - This allows us to set the bit and to reset the bit
- To calculate costs of increment operation:
  - Observe: increment only sets one bit to 1.
- Therefore:
  - n increments cost at most $2n$ bit-flips

```
000000
000001
000010
000011
000100
000101
000110
000111
001000
001001
001010
001011
001100
001101
001110
001111
010000
010001
010010
…
```

Potential Method

- Represent charges as potential energy
  - Potential function Φ maps each state of the data structure to a number
  - Amortized cost of an operation:
    - actual cost + change in potential
  - Implies:
    - Amortized costs of *n* operations
      - actual costs of *n* operations + change in potential

Potential Method

- Stack:
  - Potential = number of elements in stack
    - Potential can never be less than zero
  - Amortized cost of a stack operation
    - Push: cost plus change in potential = 1 + 1
    - Multipop / pop: cost plus change in potential = k - k

Potential Method

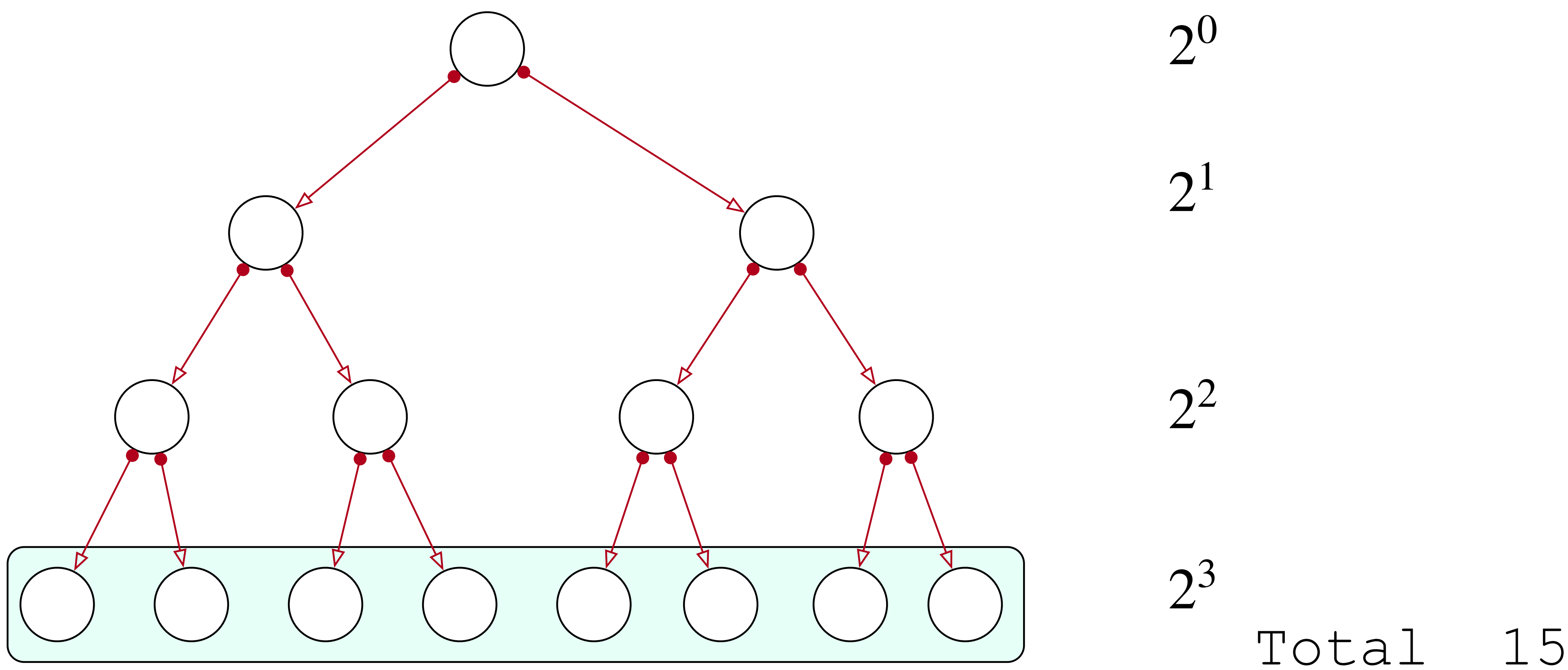
- Counter:
  - Potential = number of bits set (= number of one bits)
  - Amortized cost of an increment:
    - If i^{th} increment resets t_i bits:
      - Cost is $1 + t_i$
      - Amortized cost is
        - cost + potential change
          $\leq (t_i + 1) + (1 - t_i) = 2$

Binary Trees and Heaps

Thomas Schwarz, SJ

Behavior of Trees

- A full binary tree of depth n has
  - $1 + 2 + 2 \cdot 2 + 2 \cdot 2 \cdot 2 + \ldots + 2^{n-1}$
  - $= (111\ldots1)_2 = 2^n - 1$ places



Behavior of Trees

- Reversely:
  - To store *m* elements in a binary tree:
    - Need a tree of depth d such that
      - $2^{(d-1)} - 1 \leq m < 2^d - 1$
    - Equivalent to
      - $2^{d-1} \leq m + 1 < 2^d$
      - $d - 1 \leq \log_2(m + 1) < d$
      - $d - 1 = \lfloor \log_2(m + 1) \rfloor$

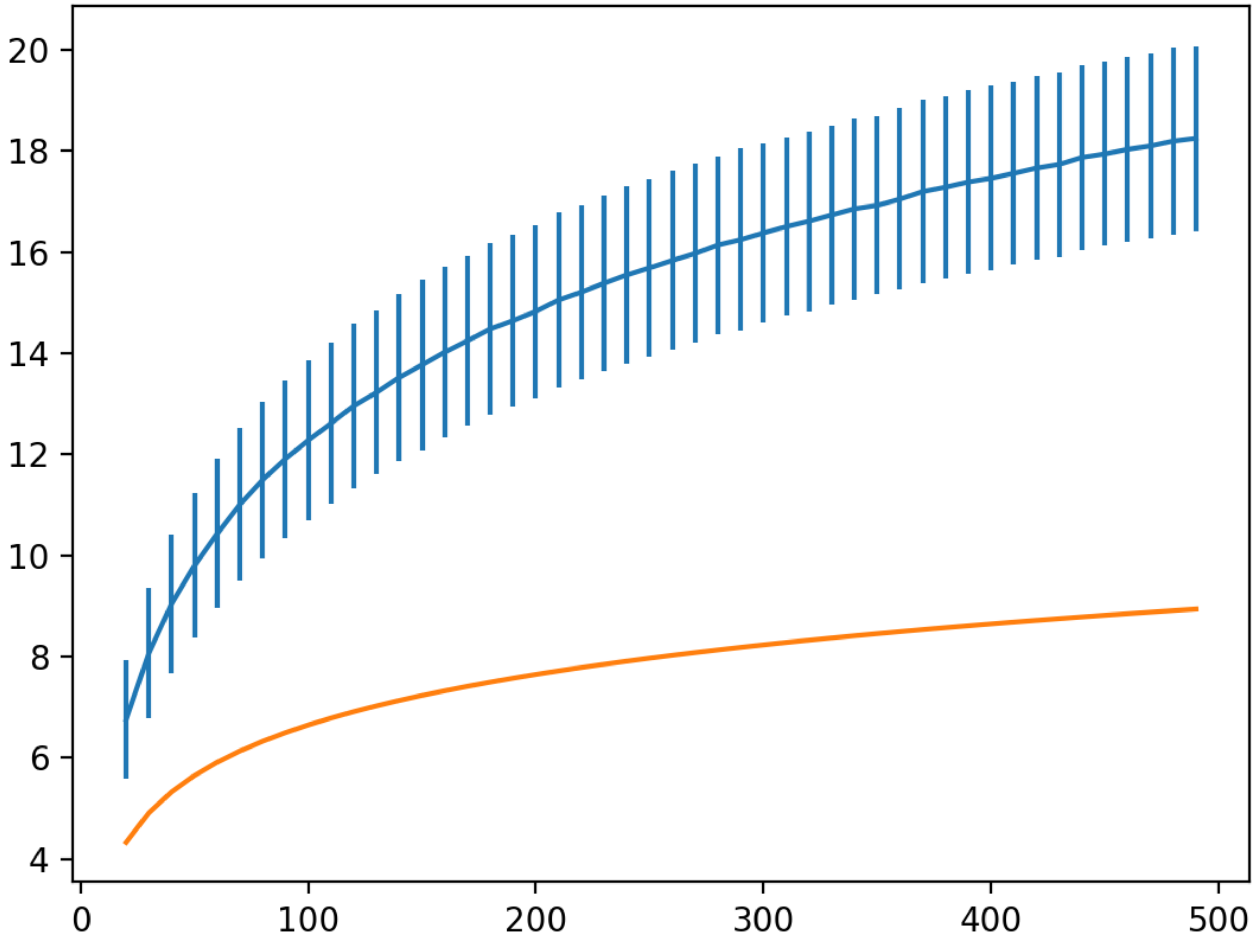
Behavior of Trees

- This parsimony is not natural
  - Random inserts: Trees have much larger depth
  - Self-modifying trees restructure themselves in order to get closer
- Importance:
  - Searching an element takes time ~ to depth
  - Inserting an element takes time ~ to depth

Behavior of Trees

- Experiment:
  - Insert *n* elements into a binary tree
  - Get the depth
  - Repeat 10,000 times
  - Depict mean plus/minus standard deviation

Behavior of Trees



Behavior of Trees

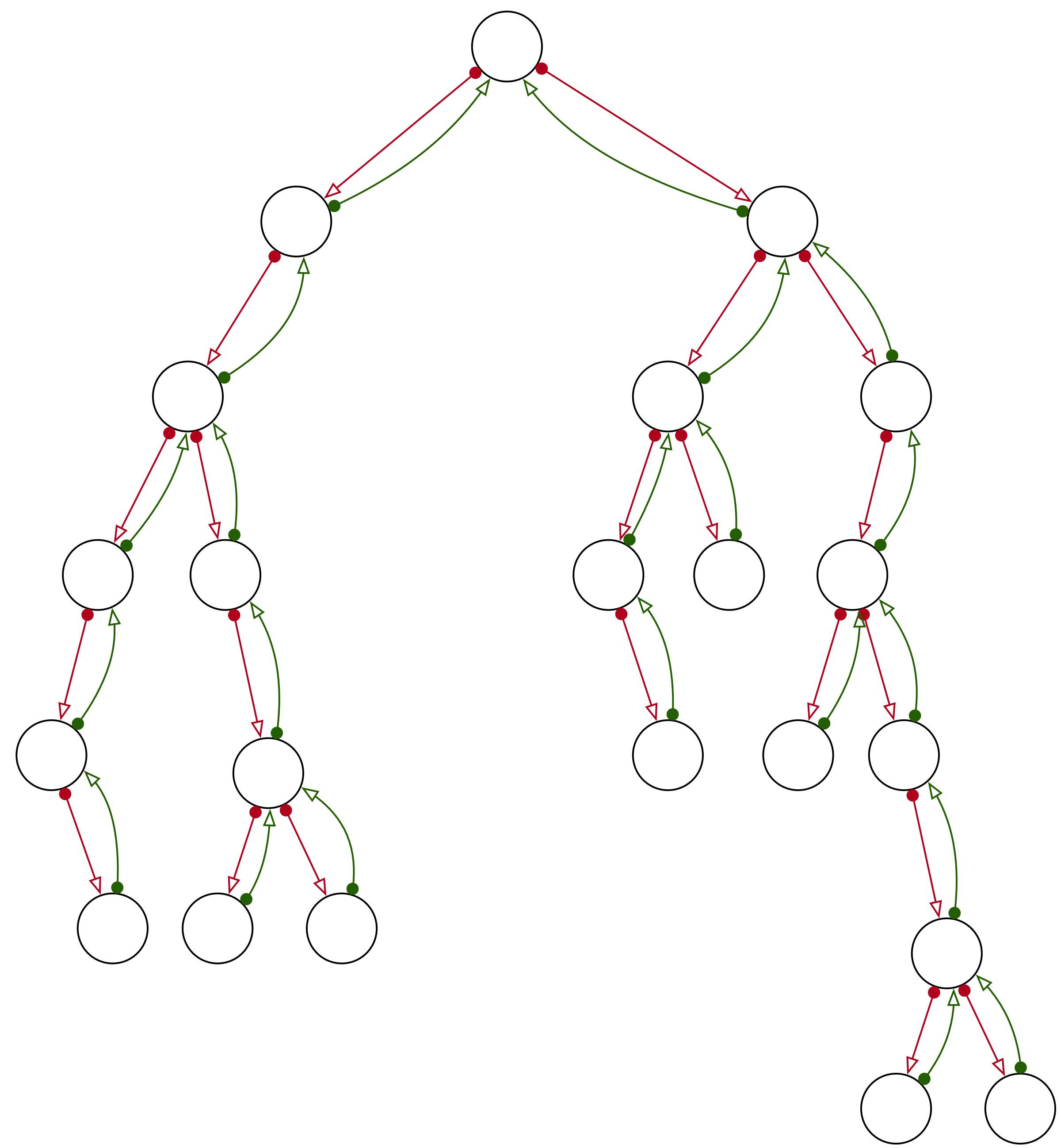
- On average
  - Trees have more than twice necessary depth
- But on average
  - Behavior is still logarithmic
- Theory: Height of a random binary search tree for a random permutation of n elements is
  - $\alpha \log_e(n)$ with $\alpha \approx 4.31107$
  - $= 2.98821 \log_2(n)$
    - Robson 1979 / Devroye 1986

Decorating Binary Search Trees

- General principle for Data Structures:
  - Can store more information in order to improve performance
- Example:
  - Removal of elements from a binary search tree
    - Difficult because we need to find parent
    - Can be made simpler by having a parent pointer

Binary Trees with Parent Link

- Each node stores a link to the parent
- For root, link is None
  - Faster deletes at the cost of more storage per node



Binary Trees with Parent Link

- Expand to a key-value store by adding a field for record
- Add a parent link

```
class Node:
    def __init__(self, value, record):
        self.value = value
        self.record = record
        self.up, self.left, self.right = None, None, None

    def __repr__(self):
        return "Node : {}, Value: {}, Record: {},
                 Left: {}, Right: {}, Up: {}".format(
                    hex(id(self)), self.value, self.record,
                    hex(id(self.left)), hex(id(self.right)),
                    hex(id(self.up)))
```

Binary Trees with Parent Link

- We have to maintain the up link:

```
def insert(self, value, record):
        new_node = Node(value, record)
        if not self.root:
            self.root = new_node
        else:
            current = self.root
            while True:
                if value < current.value:
                    if current.left:
                        current = current.left
                    else:
                        current.left = new_node
                        new_node.up = current
                        return
```

Binary Trees with Parent Link

- But deleting a record is still not trivial
  - Special case when
    - the tree is empty

```
def remove(self, value):
    if not self.root:
        return False
```

Binary Trees with Parent Link

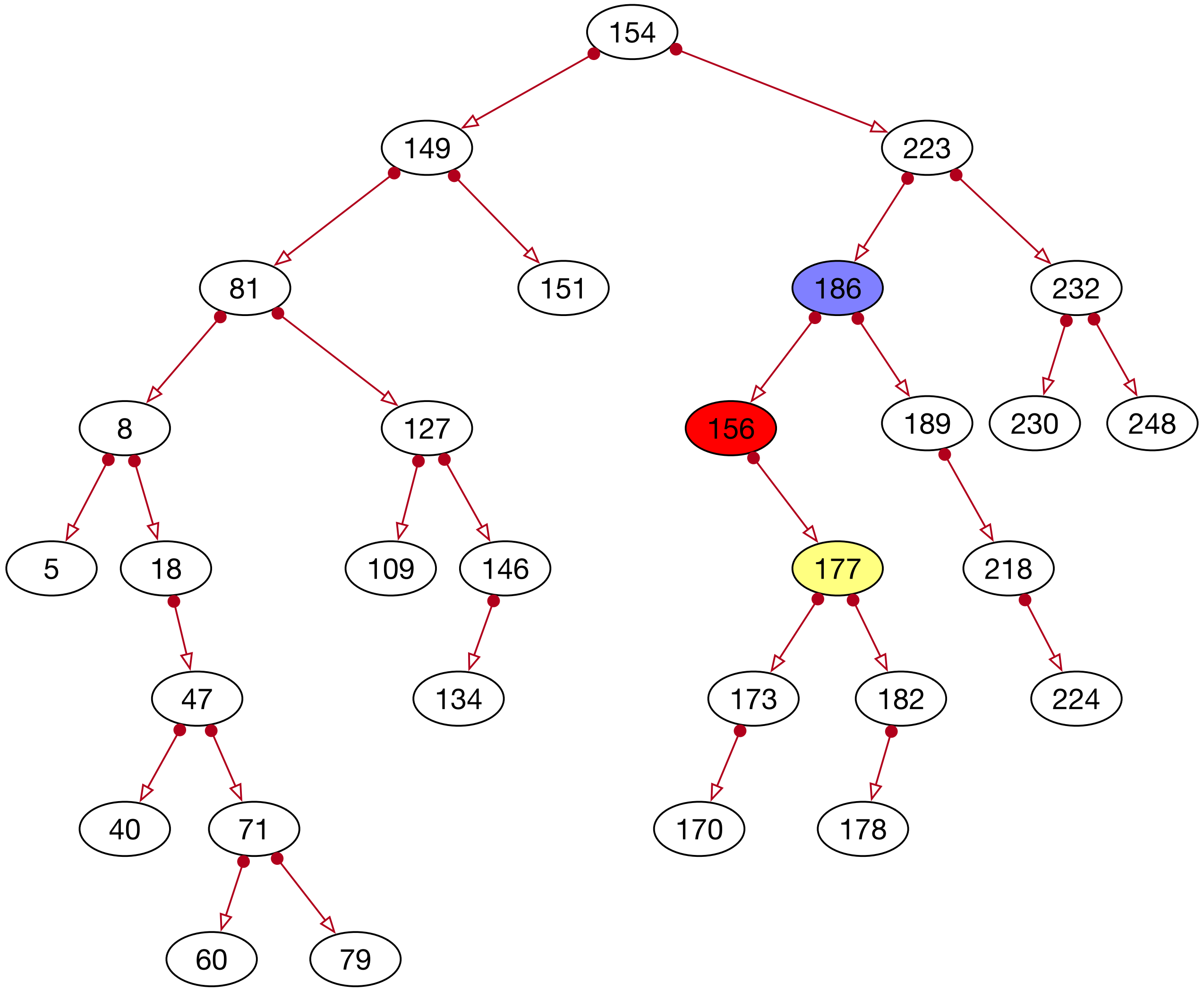
```
def remove(self, value):
        if not self.root:
            return False
        current = self.root
        while True:
            if not current:
                return False
            if value == current.value:
                break
            if value < current.value:
                current = current.left
            else:
                current = current.right
            if current == None:
                return False
        to_delete = current
```

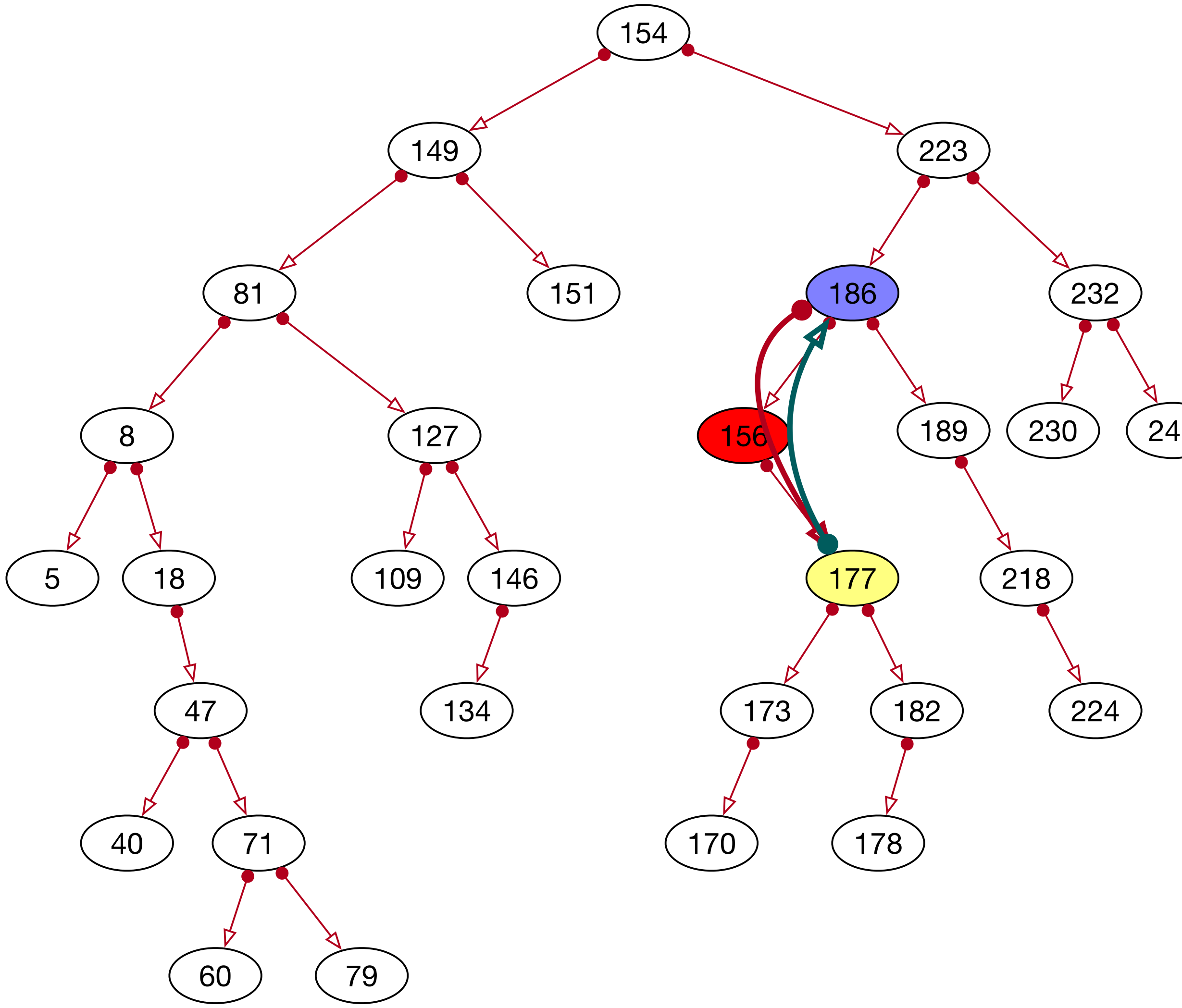
Binary Trees with Parent Link

- We still need to make additional case distinctions
  - But we no longer need a stack to keep track of the nodes
  - Case distinctions:
    - No children:
      - Just delete (unless we are deleting the root)
    - One child
    - Two children

Binary Trees with Parent Link

- Removing node with one child
- Move child up and reset **two** links





Binary Trees with Parent Link

- Special case if parent is root

```
elif not to_delete.left and to_delete.right:
            # node has only a right child
            parent = to_delete.up
            if not parent:
                self.root = to_delete.right
                return True
            else:
                if parent.left == to_delete:
                    parent.left = to_delete.right
                    to_delete.right.up = parent
                else:
                    parent.right = to_delete.right
                    to_delete.right.up = parent
                return True
```

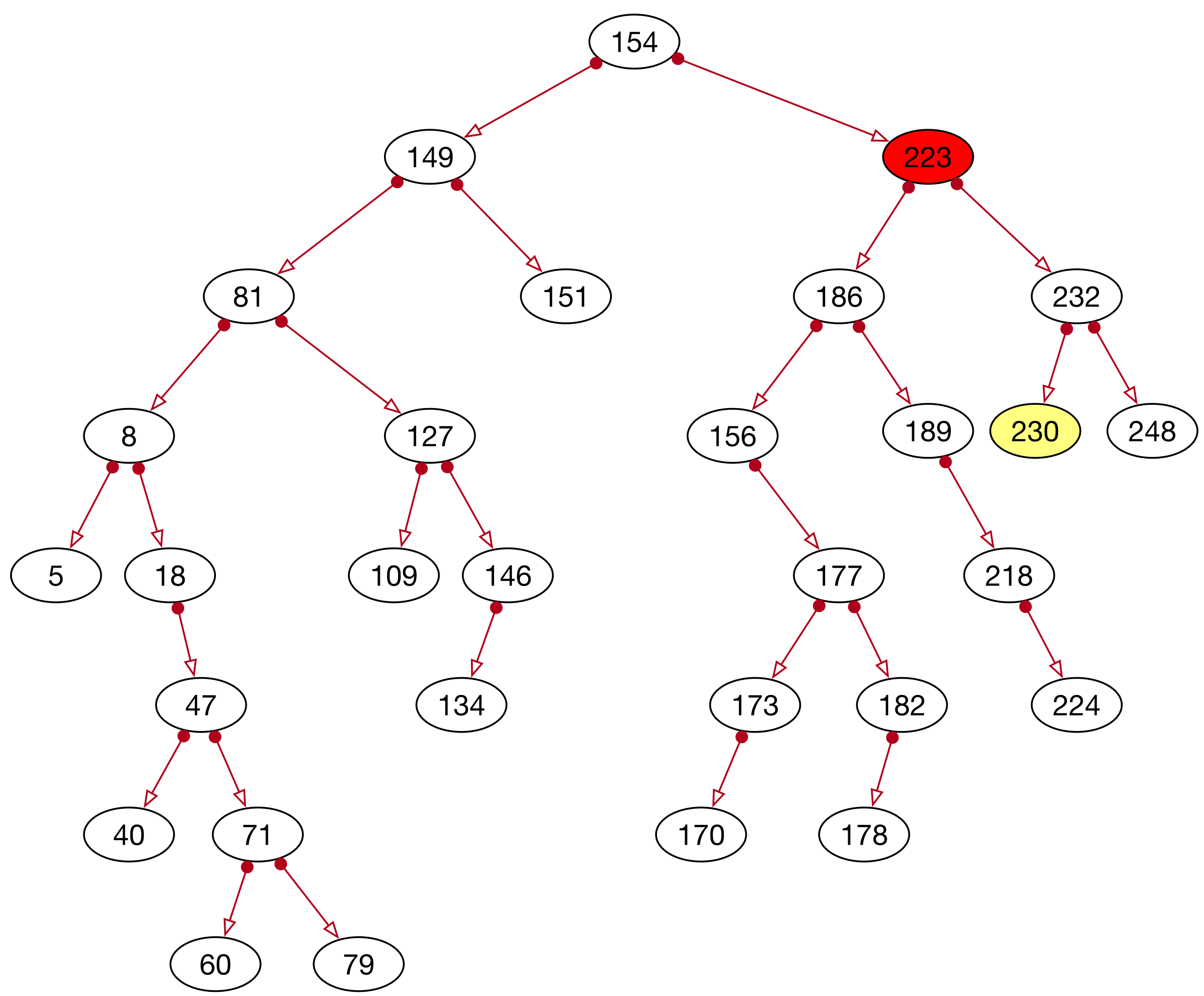
Binary Trees with Parent Link

- Otherwise: reset two links

```
elif not to_delete.left and to_delete.right:
            # node has only a right child
            parent = to_delete.up
            if not parent:
                self.root = to_delete.right
                return True
            else:
                if parent.left == to_delete:
                    parent.left = to_delete.right
                    to_delete.right.up = parent
                else:
                    parent.right = to_delete
                    to_delete.right.up = parent
                return True
```

Binary Trees with Parent Link

- Two children:
    - Identify the next node in-order traversal



Binary Trees with Parent Link

- Two children:
  - Find the next node in in-order traversal:
    - Go to the right: `current.right`
    - Then go always to the left

```
def min_value_node(a_node):
    current = a_node
    while current.left:
        current = current.left
    return current
```

Binary Trees with Parent Link

- Two nodes

```
elif to_delete.left and to_delete.right:
            #node has two children
            leaf = Binary_Tree.min_value_node(
                                         to_delete.right)
            save_value = leaf.value
            save_record = leaf.record
            self.remove(leaf.value)
            to_delete.value = save_value
            to_delete.record = save_record
```

Binary Trees with Parent Link

- Safe the values of the resulting leaf

```
elif to_delete.left and to_delete.right:
            #node has two children
            leaf =
             Binary_Tree.min_value_node(to_delete.right)
            print('leaf',leaf)
            save_value = leaf.value
            save_record = leaf.record
            self.remove(leaf.value)
            to_delete.value = save_value
            to_delete.record = save_record
```

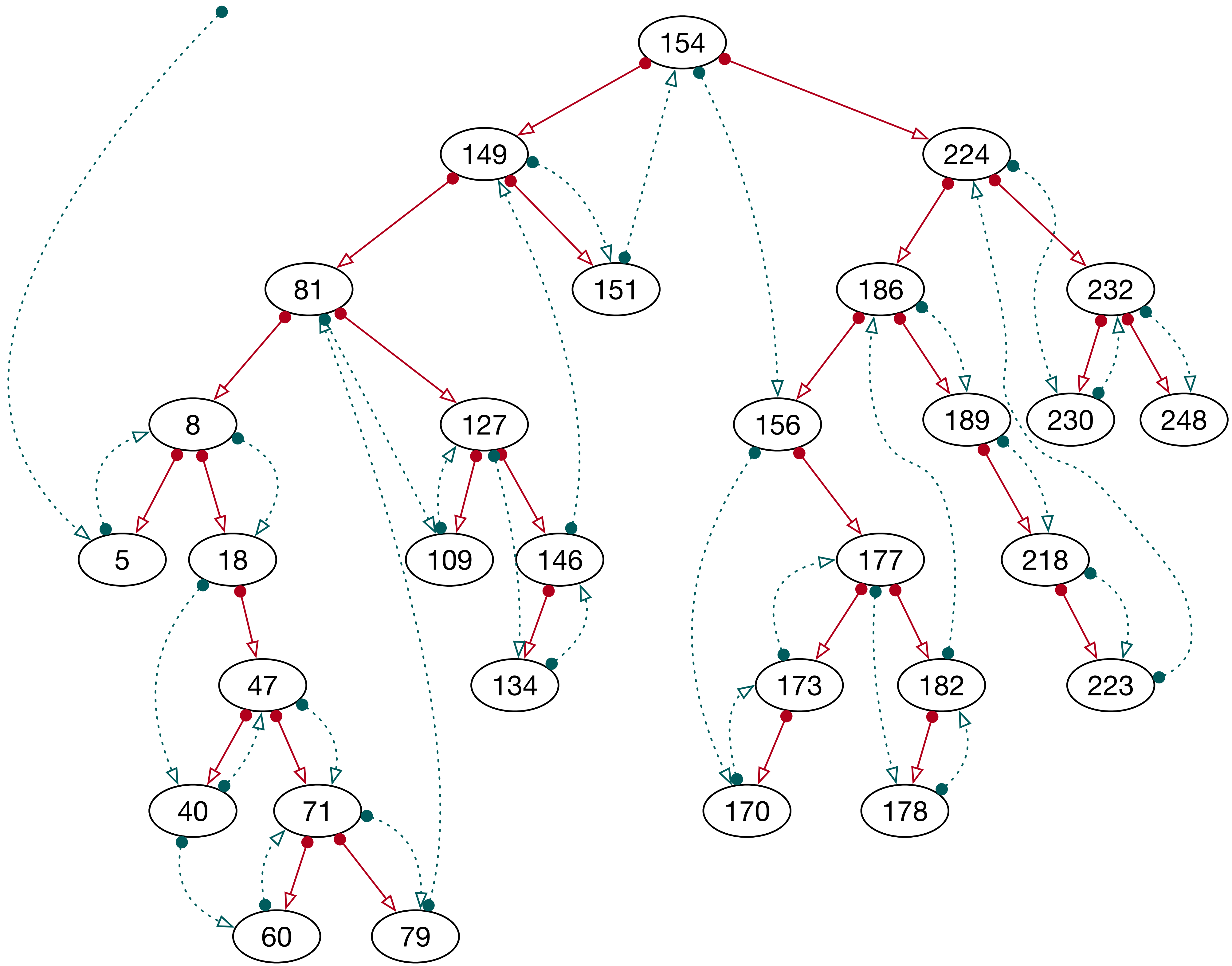
Binary Trees with Parent Link

- Then delete the leaf
  - I cheat by using recursion

```
elif to_delete.left and to_delete.right:
            #node has two children
            leaf =
             Binary_Tree.min_value_node(to_delete.right)
            print('leaf',leaf)
            save_value = leaf.value
            save_record = leaf.record
            self.remove(leaf.value)
            to_delete.value = save_value
            to_delete.record = save_record
```

Binary Trees with Parent Link

- Non-recursive in-order traversal
  - Here is a tree with an additional set of links for in-order traversal


154
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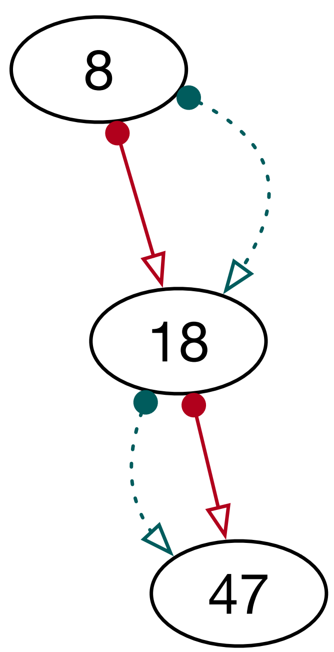
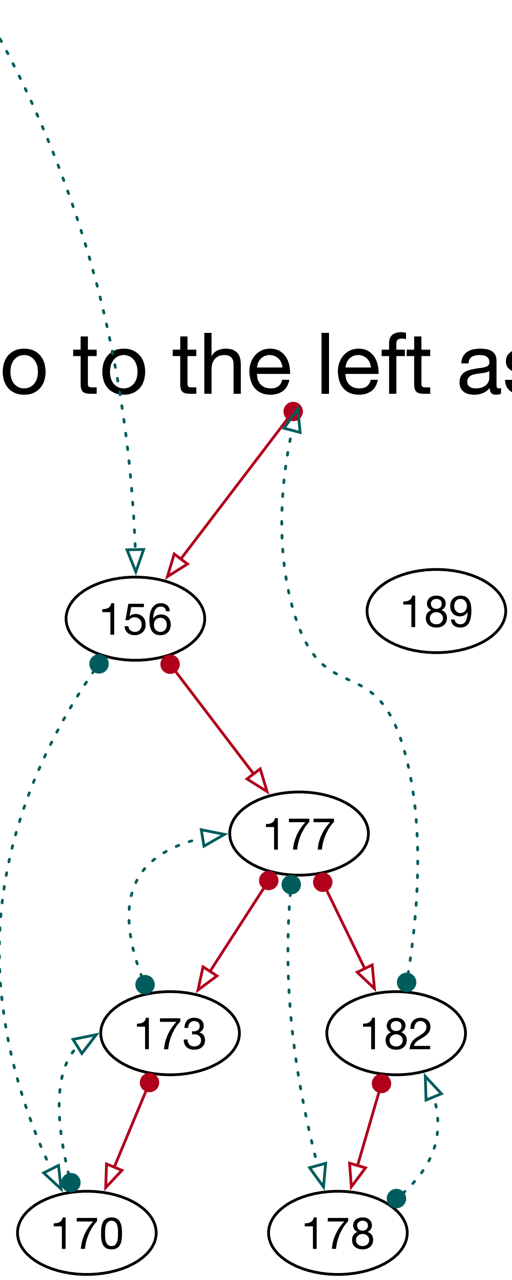
230
248
5
18
109
146
177
218
47
134
173
182
223
40
71
170
178
60
79

Binary Trees with Parent Link

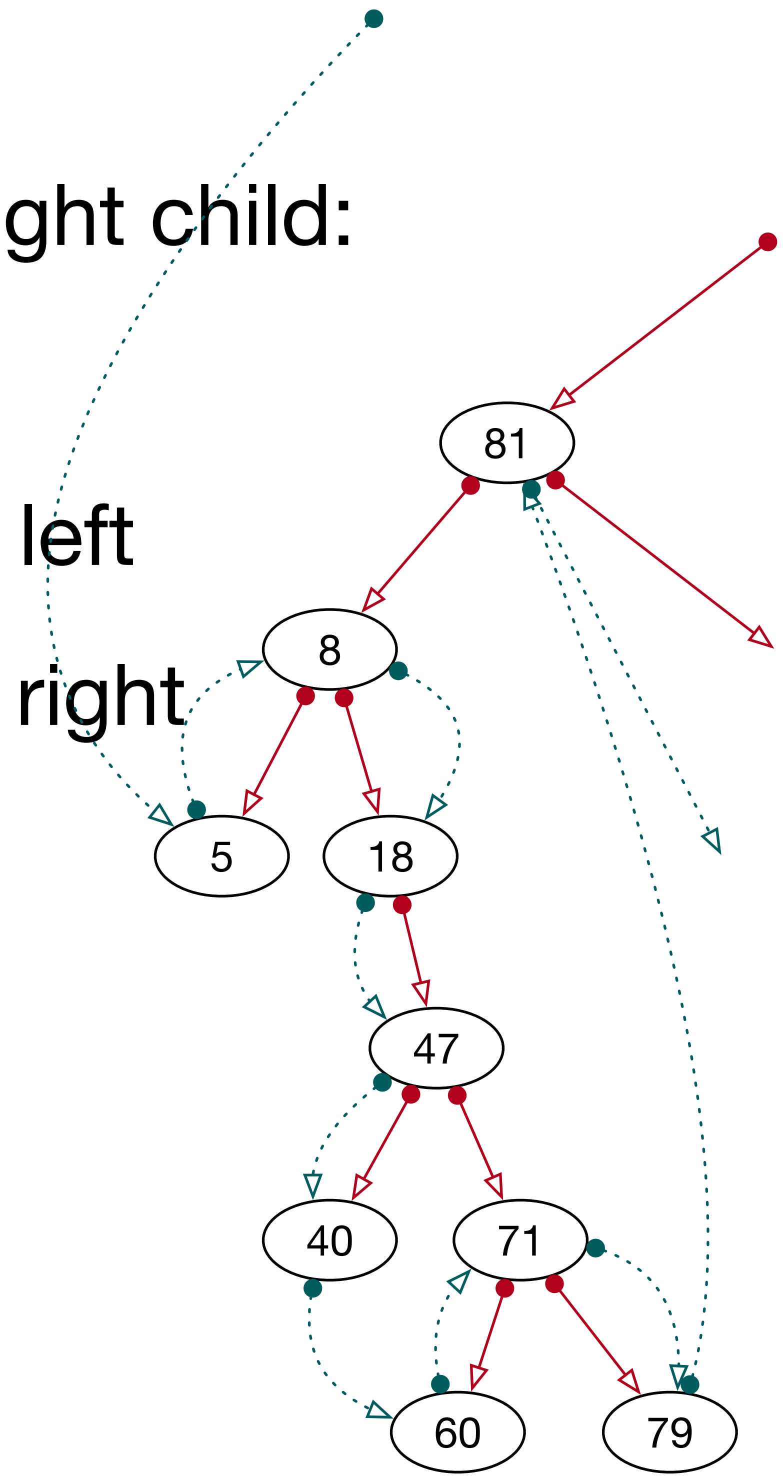
- What is the next node:
  - If the node has a right child:
    - Go one to the right, then go to the left as much as possible





Binary Trees with Parent Link

- What is the next node if there is no right child:
    - If parent is to the left:
        - Follow parents if they are to the left
        - Then take the first parent to the right



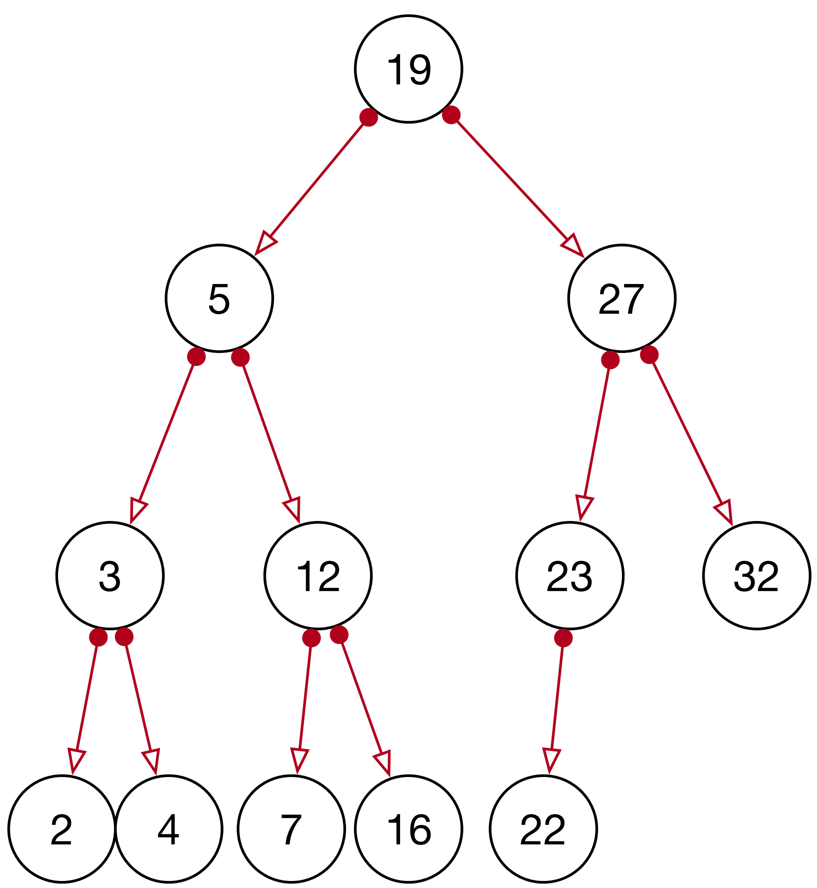
Binary Trees with Parent Link

- Thus:
  - Can do in-order traversal without a stack or recursion

Binary Trees using Arrays

Using Arrays

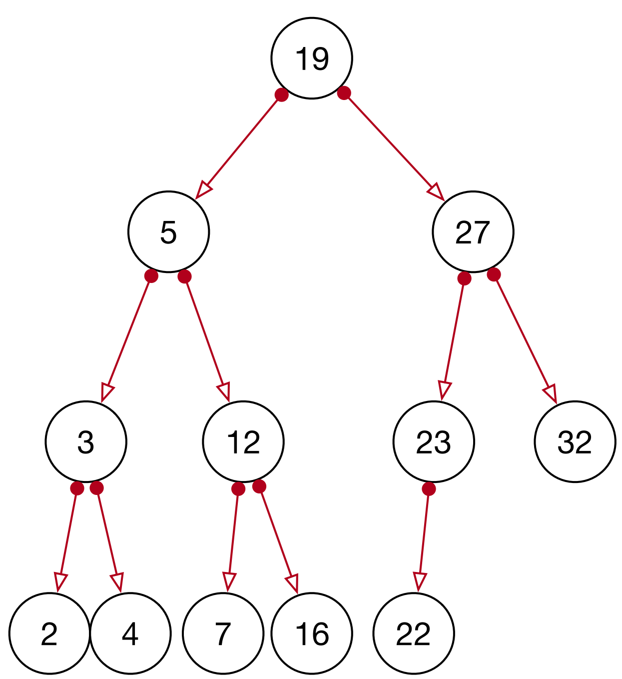
- In a tree, each node has up to two children
  - Can organize nodes in an array
    - Leave first spot open



—	19	5	27	3	12	23	32	2	4	7	16	22
	1	2	3	4	5	6	7	8	9	10	11	12

Using Arrays

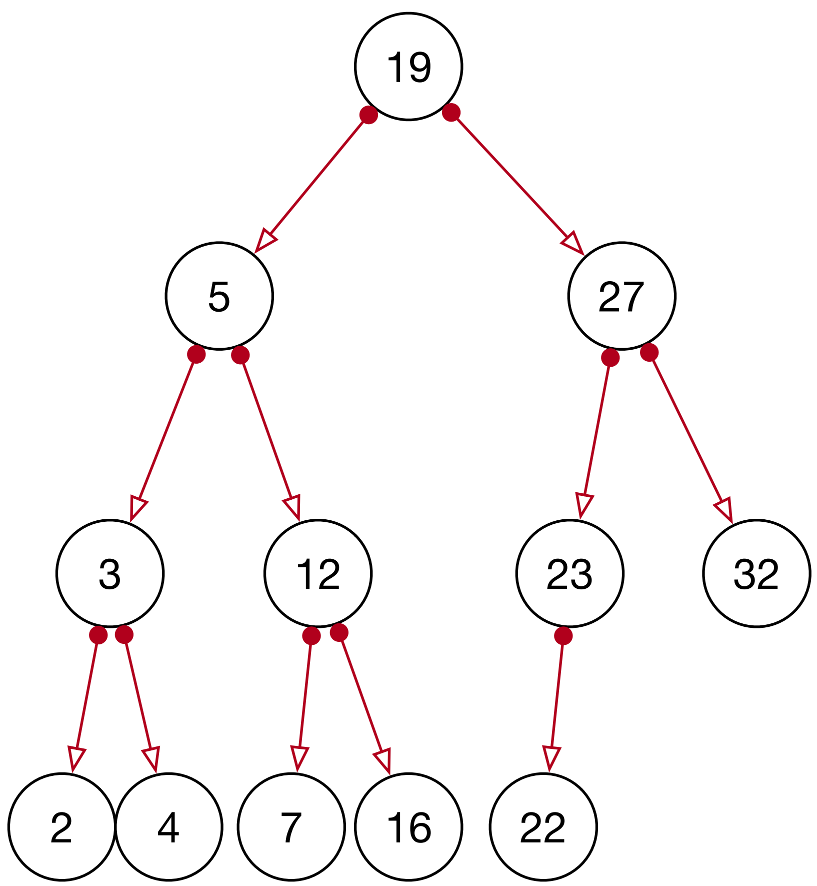
- Left child of node at index i
    - Located at index $2i$
- Right child of node at index i
    - Located at index $2i + 1$



–	19	5	27	3	12	23	32	2	4	7	16	22
	1	2	3	4	5	6	7	8	9	10	11	12

Using Arrays

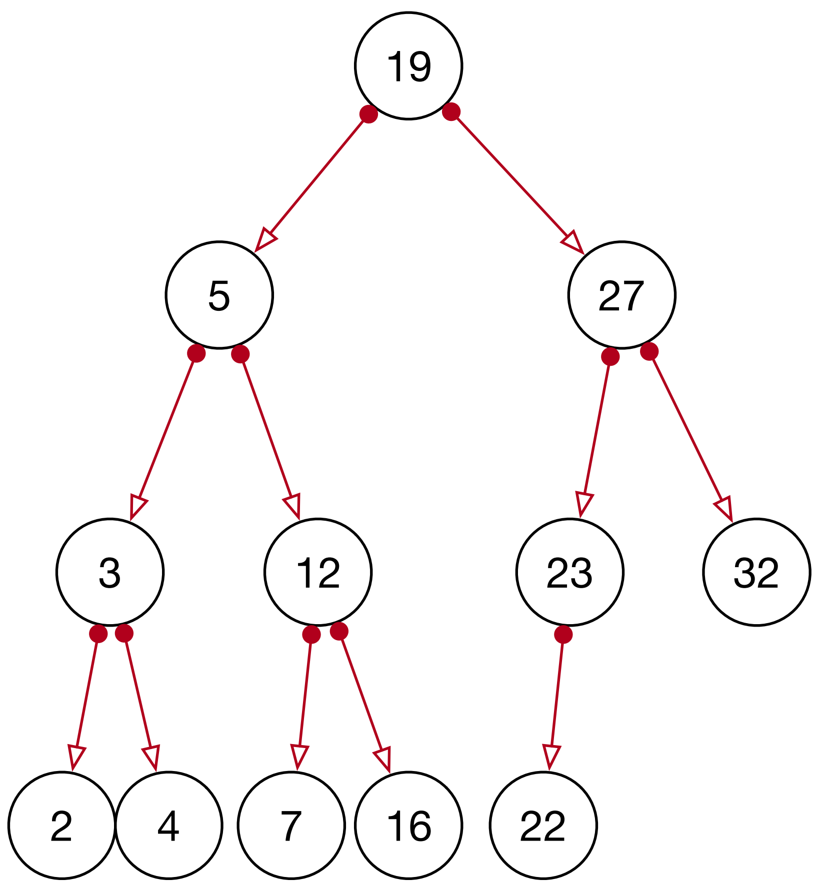
- Parent of node at index i is located at index $i//2$
  - Mathematical notation: $\lfloor \frac{i}{2} \rfloor$



—	19	5	27	3	12	23	32	2	4	7	16	22
	1	2	3	4	5	6	7	8	9	10	11	12

Using Arrays

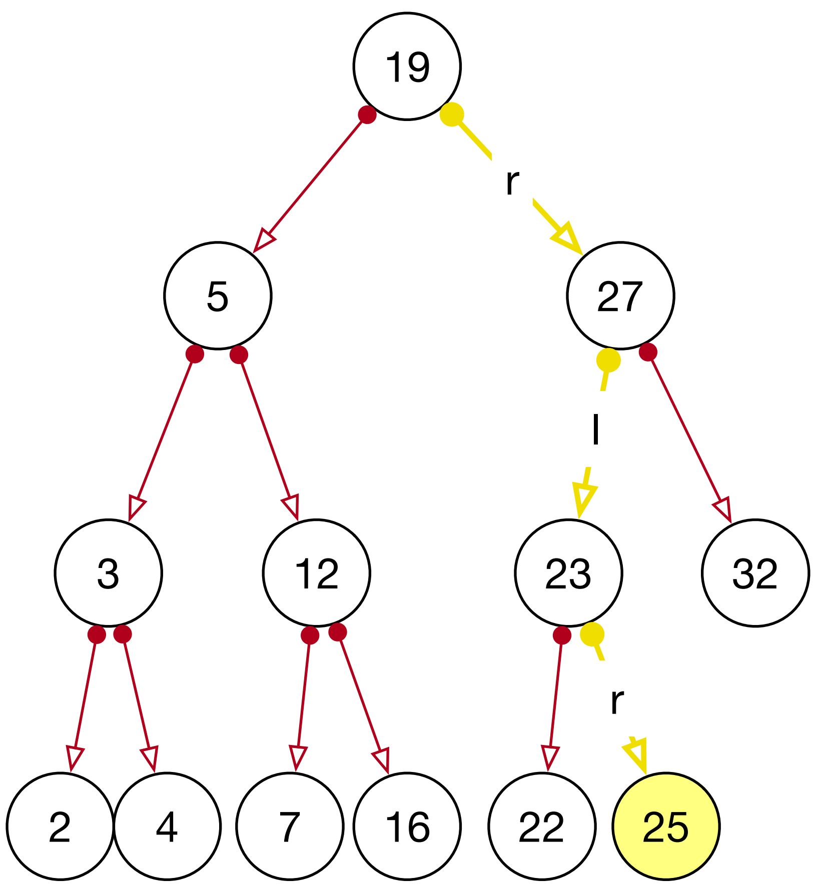
- Right children are at odd indices, left children are even indices



—	19	5	27	3	12	23	32	2	4	7	16	22
	1	2	3	4	5	6	7	8	9	10	11	12

Using Arrays

- We can calculate the index if we are given a sequence of directions

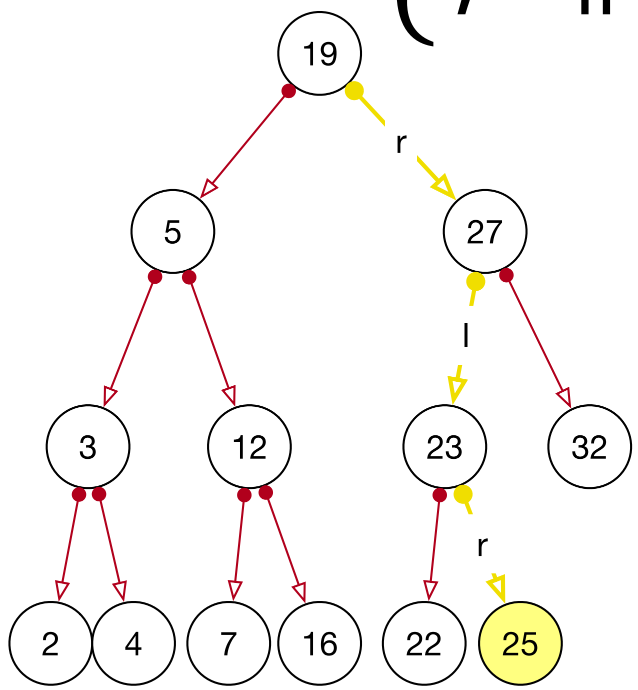


—	19	5	27	3	12	23	32	2	4	7	16	22	25
	1	2	3	4	5	6	7	8	9	10	11	12	13

rlr ((1*2+1)*2)*2+1 =13

Using Arrays

- Define $r(n) := 2n + 1, l(n) := 2n$
- Then node is at index $(o_m \circ o_{m-1} \circ \ldots \circ o_2 \circ o_1)(1)$
- where $o_i = \begin{cases} l & \text{if we go left in step } i \\ r & \text{if we go right in step } i \end{cases}$



—	19	5	27	3	12	23	32	2	4	7	16	22	25
	1	2	3	4	5	6	7	8	9	10	11	12	13

rlr ((1*2+1)*2)*2+1 =13

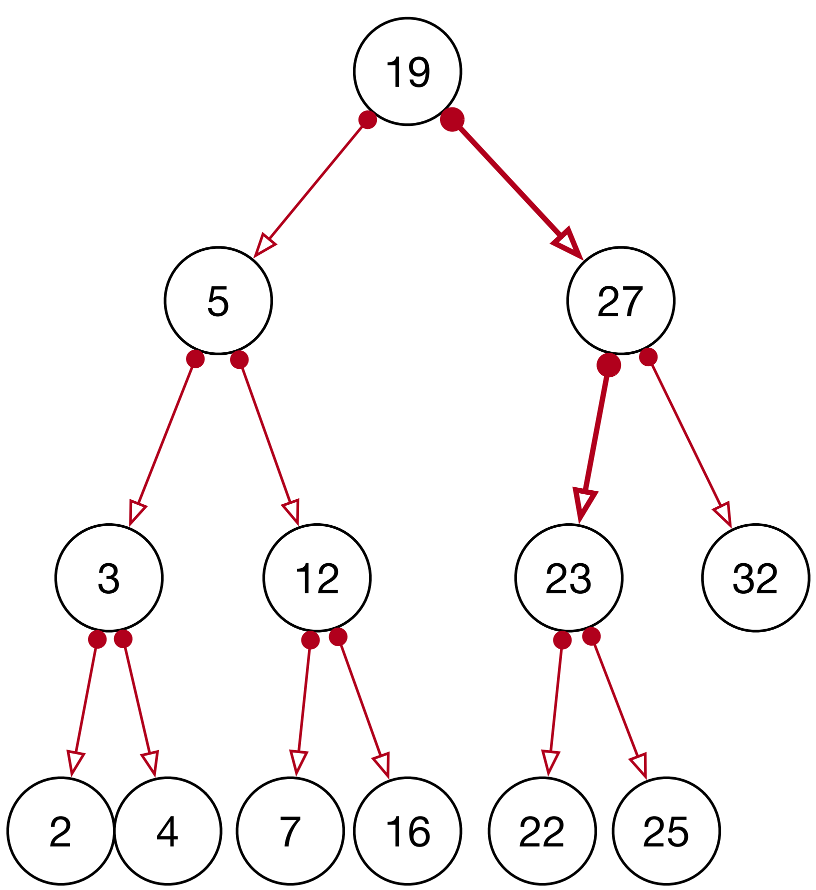
$\boxed{r \circ l \circ r(1)}$

Using Arrays

- Can we do something about the unused first element in the array?
    - We just need to adjust the index: by adding 1 and subtracting 1

Using Arrays

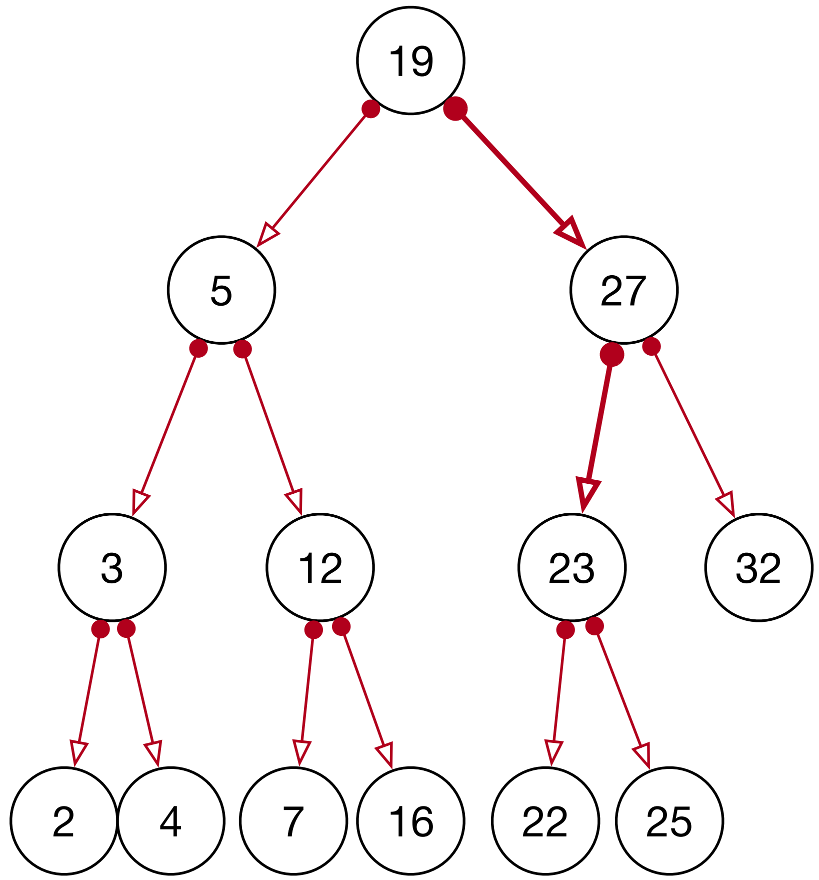
- Children of node i are now $2 \cdot (i+1) - 1 = 2 \cdot i + 1$ and $(2 \cdot (i+1) + 1) - 1 = 2 \cdot i + 2$



19	5	27	3	12	23	32	2	4	7	16	22	25
0	1	2	3	4	5	6	7	8	9	10	11	12

Using Arrays

- Parent of a node located at index i is located
    - at index $\lfloor \frac{i+1}{2} \rfloor - 1$



19	5	27	3	12	23	32	2	4	7	16	22	25
0	1	2	3	4	5	6	7	8	9	10	11	12

Using Arrays

- One advantage:
  - We automatically have a way to find the parent

Priority Queue

- ADS with
  - Insertion
  - Popping maximum element
- Example: insert 5, insert 4, insert 10, pop, insert 7, insert 3, pop, insert 2, pop, pop
  - Returns on insert 5, insert 4, insert 10, **pop**, insert 7, insert 3, pop, insert 2, pop, pop: 10
  - Returns on insert 5, insert 4, insert 10, pop, insert 7, insert 3, **pop**, insert 2, pop, pop: 7
  - Returns on insert 5, insert 4, insert 10, pop, insert 7, insert 3, pop, insert 2, **pop**, pop: 5
  - Returns on insert 5, insert 4, insert 10, pop, insert 7, insert 3, pop, insert 2, pop, **pop**: 4

Priority Queues

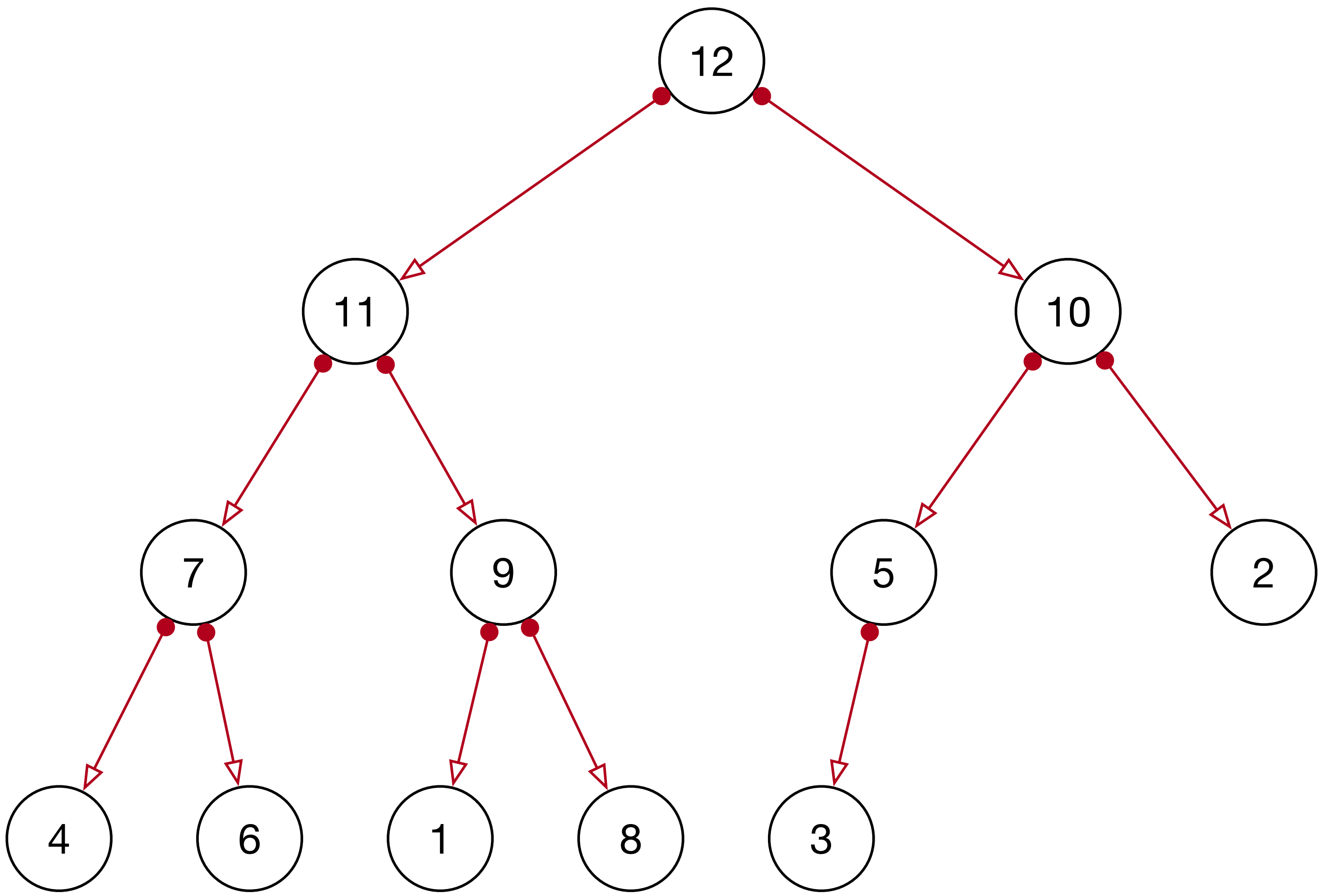
- Simplistic implementation
  - A list
    - Whenever we look for an element, we look for the maximum of the list
    - Run time: Proportional to the length of the list

Priority Queues

- Favorite implementation:
  - Heap:
    - A ***complete*** binary tree
      - Tree is maximum balanced
    - That is **partially** ordered

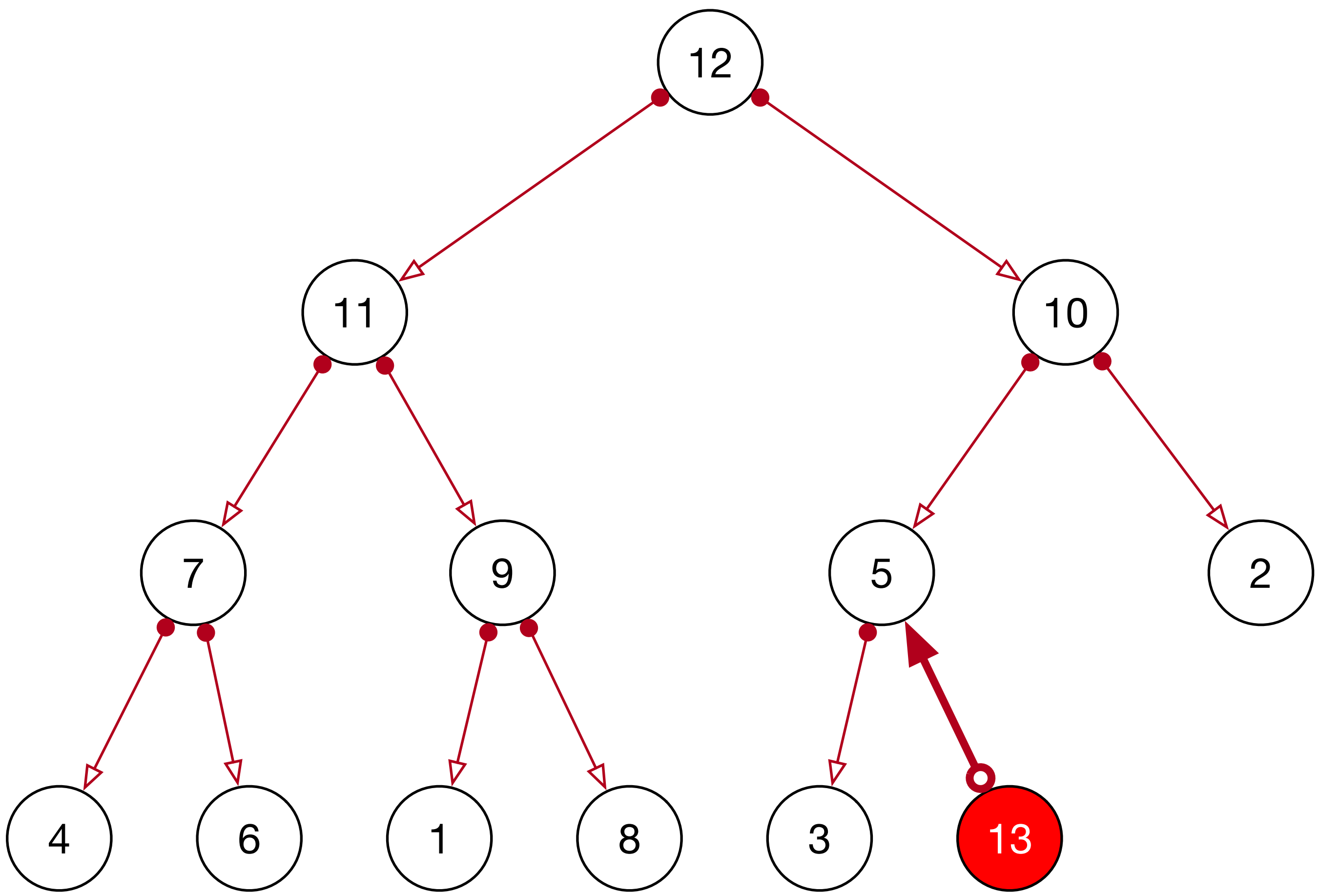
Priority Queues

- Heaps as binary tree
  - Complete:
    - No nodes missing
    - Last generation filled from left
  - Partially ordered:
    - parent has larger value than child



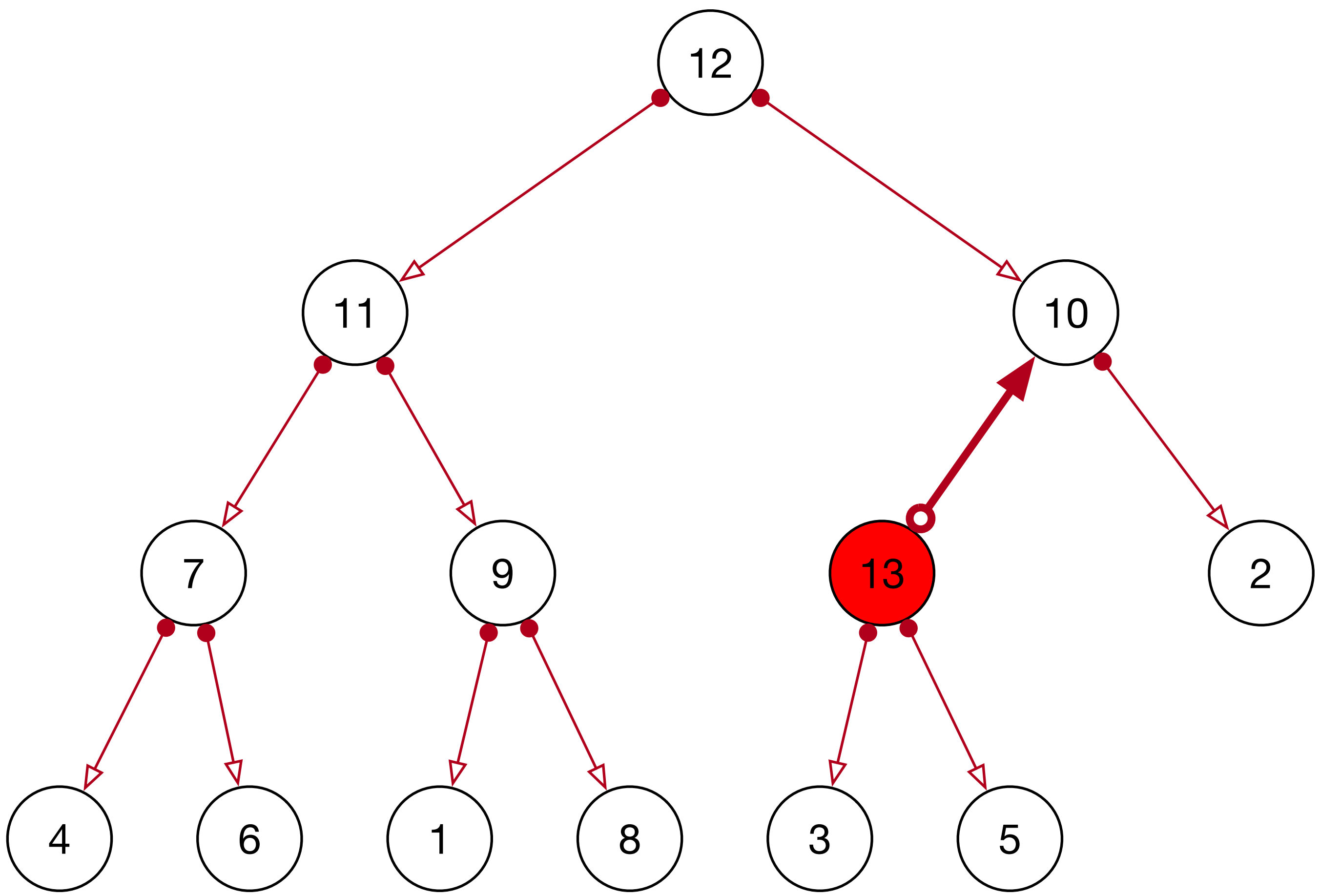
Priority Queues

- Operations: Insertion
  - Insert at the next spot
  - If the new node is larger than the parent:
    - swap with parent



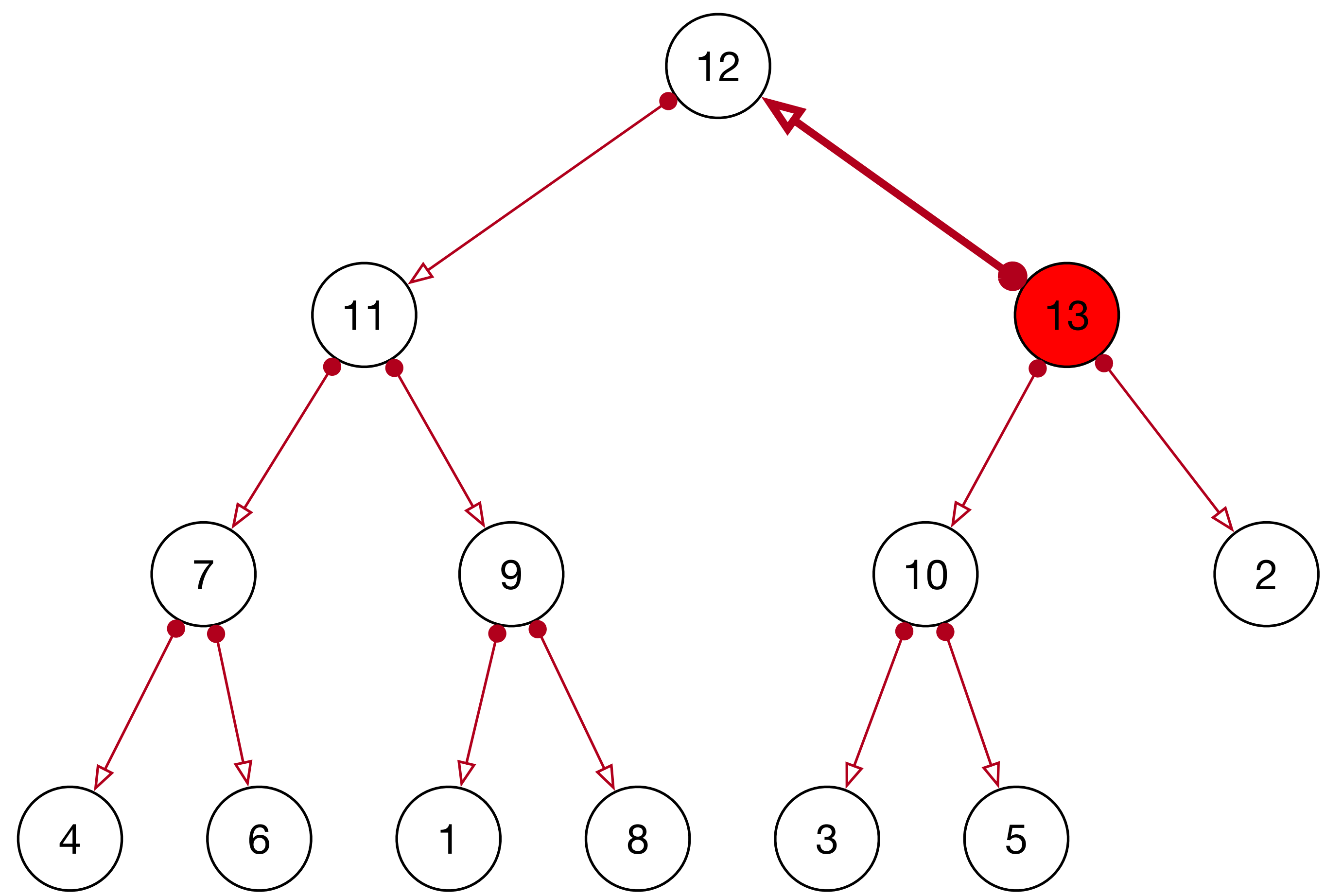
Priority Queues

- This is repeated
  - if necessary



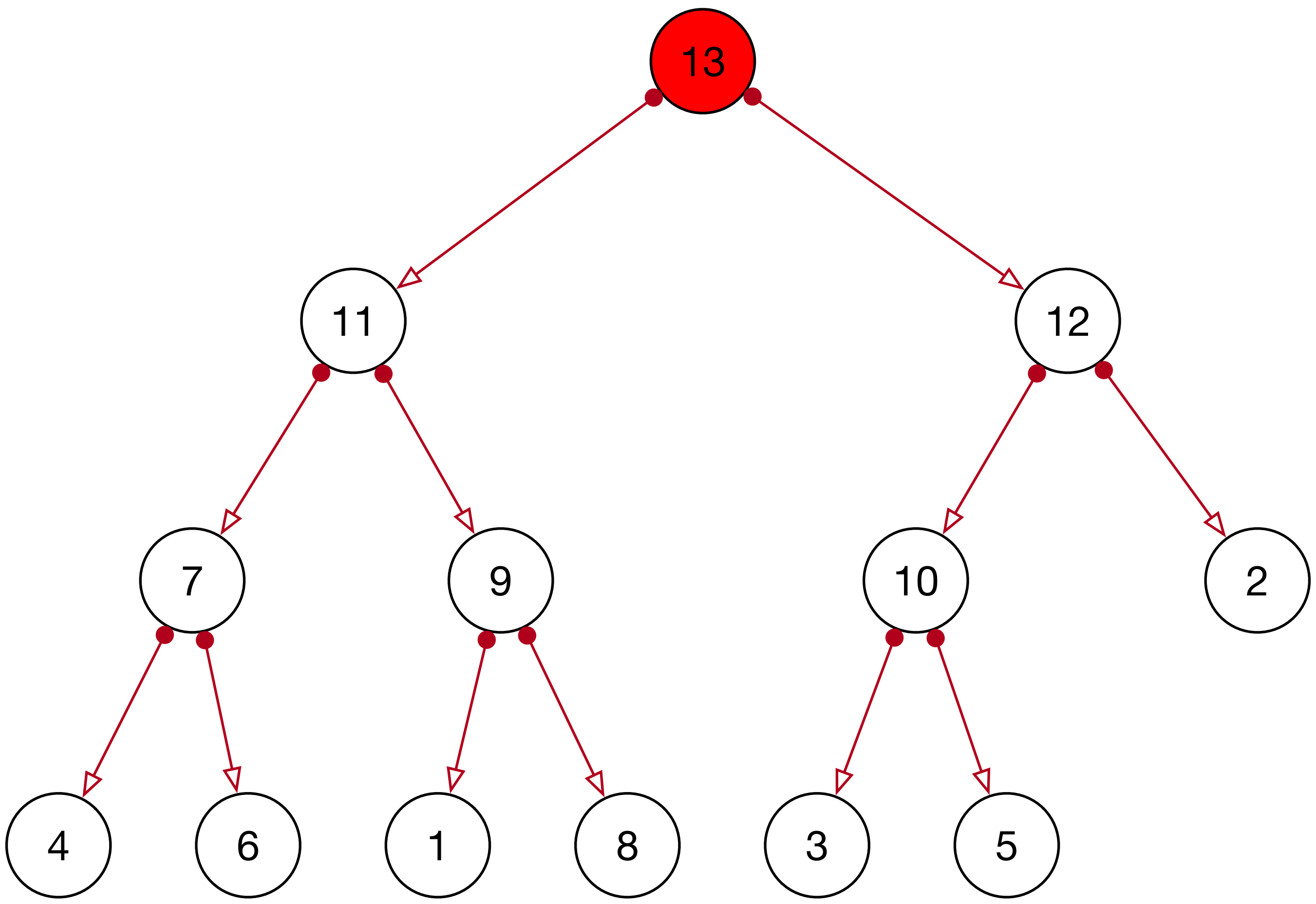
Priority Queues

- Notice:
  - The only violation of order can be with parent



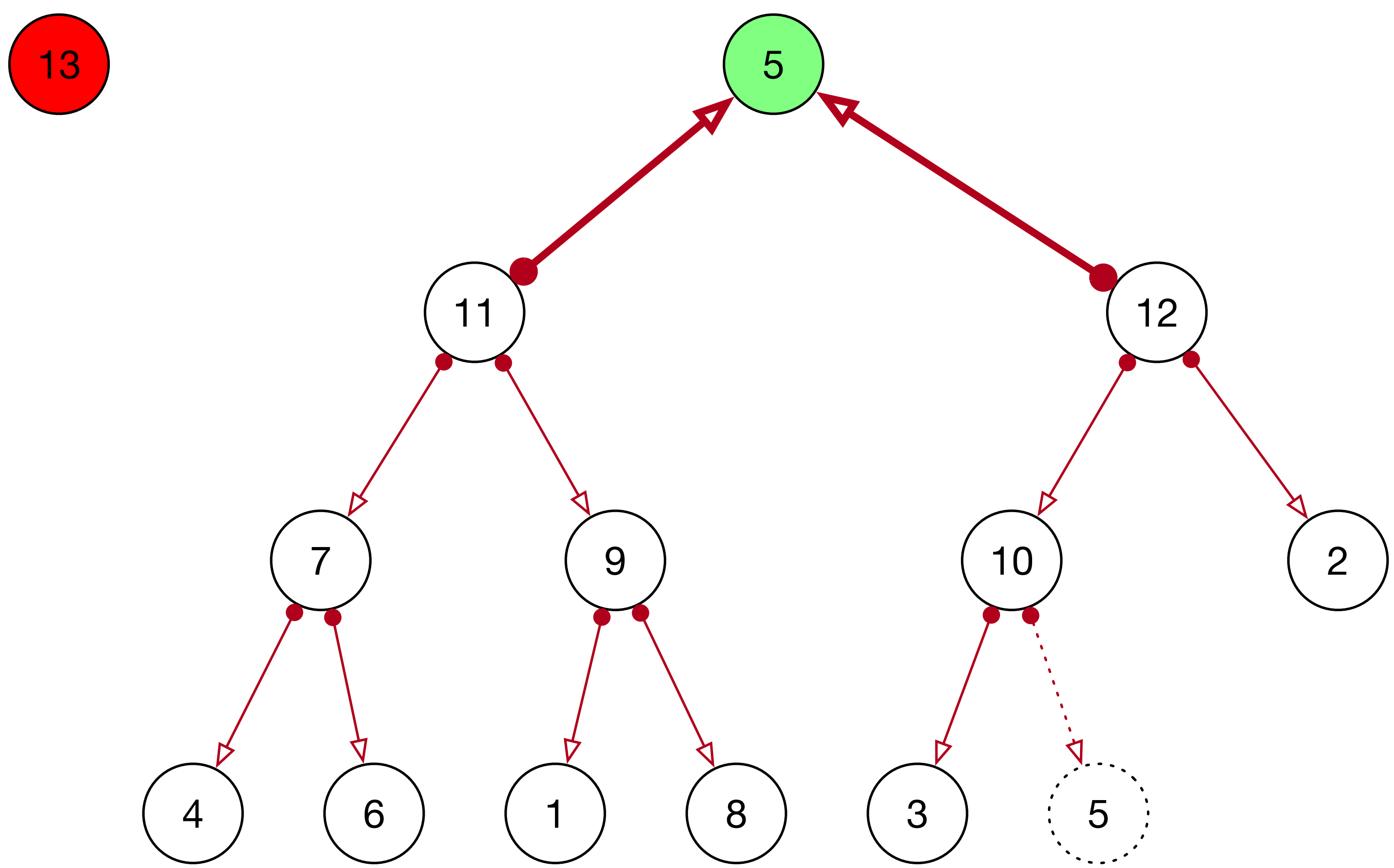
Priority Queues

- There are at most $\log_2(n)$ swaps
  - Compared to n



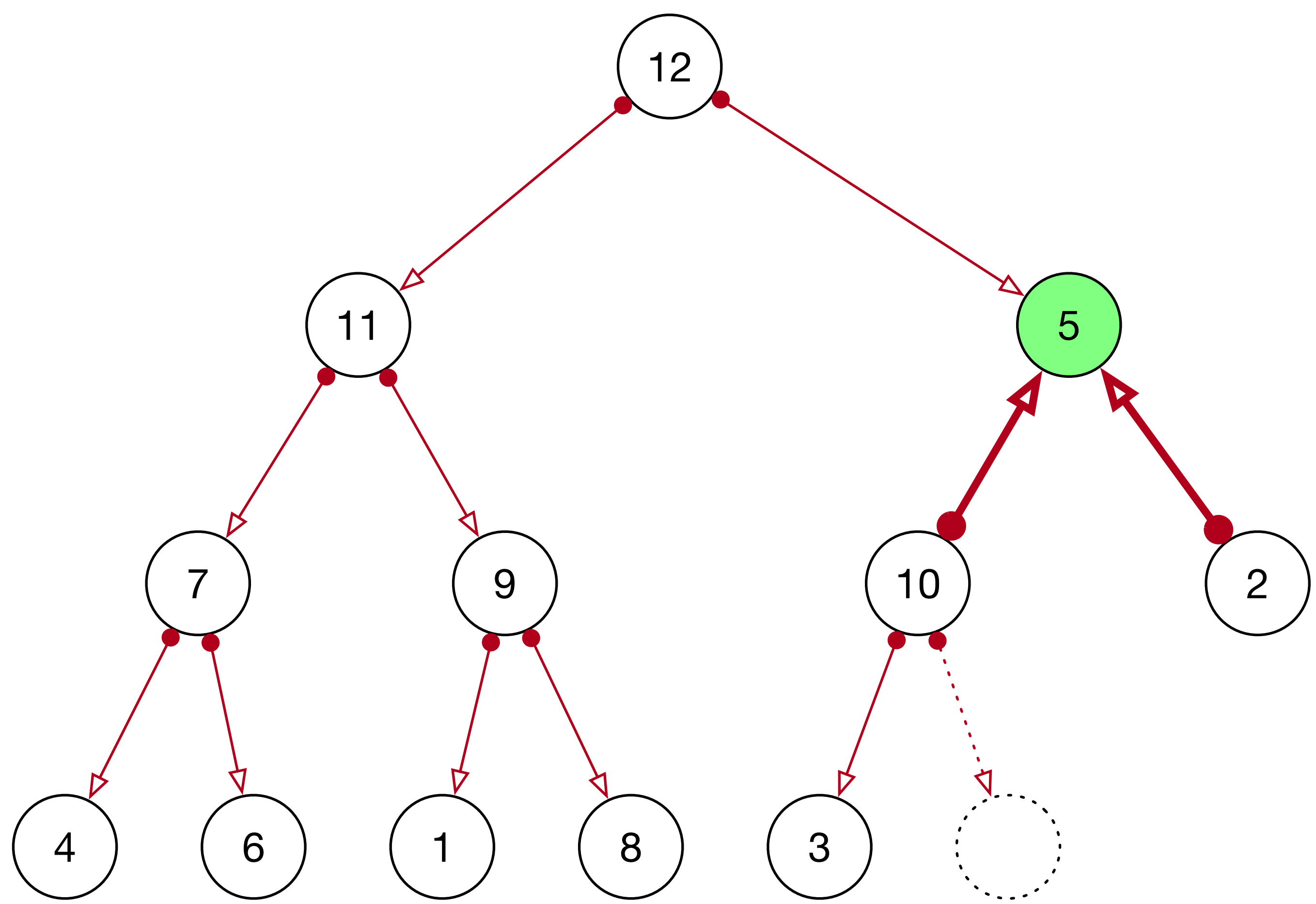
Priority Queues

- Remove Maximum:
  - Maximum is at the top, remove it
  - Move last element into the top position



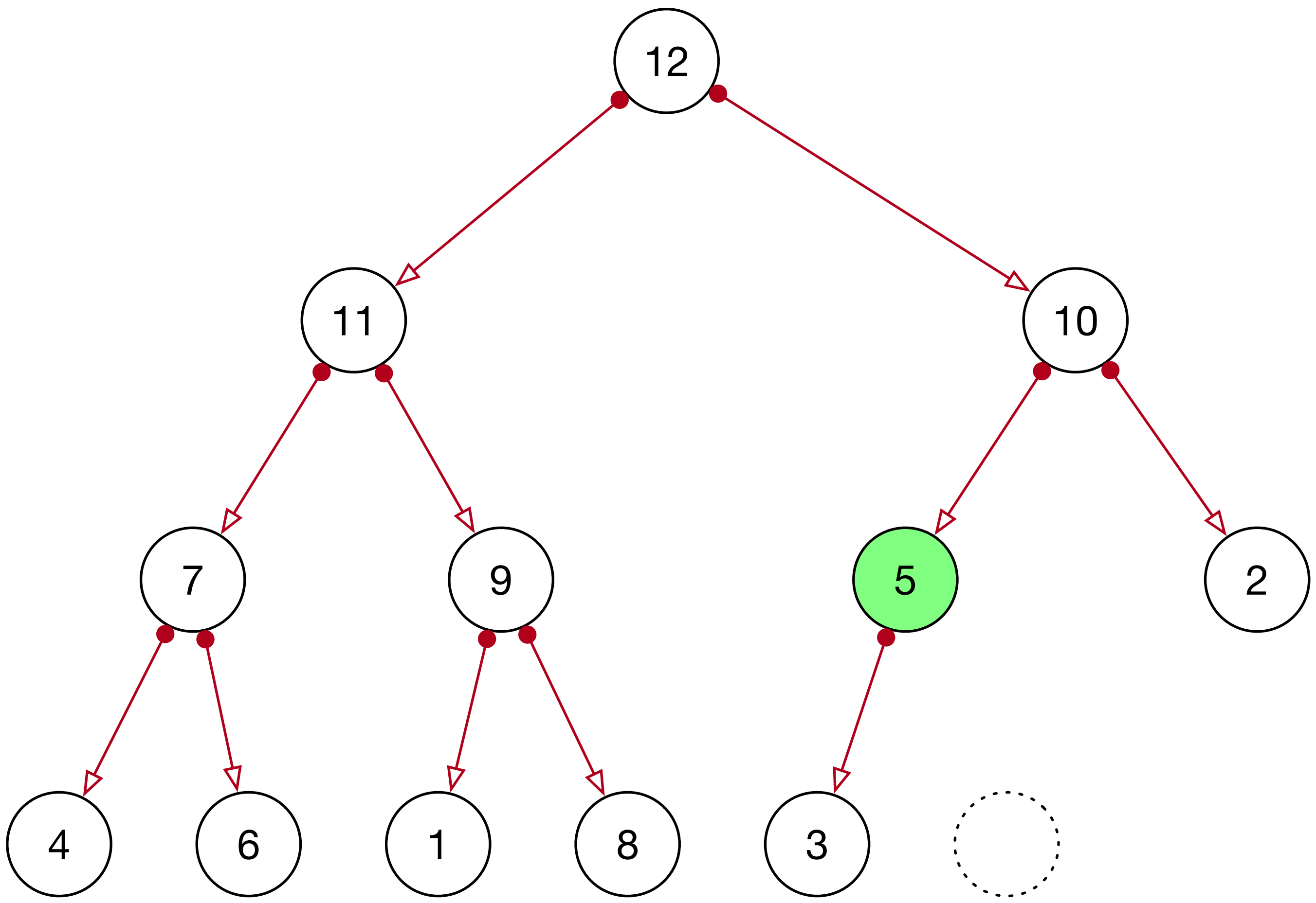
Priority Queues

- Then restore the heap property
  - Move up the ***larger*** sibling



Priority Queues

- Until there is no violation



Priority Queues

- Implementation:
    - Need to implement two "heapify" operations
        - Going up for insert
        - Going down for extract maximum

Priority Queues

- Define a class PQ with class methods for index calculation

```
class PQ:
    def __init__(self):
        self.array = []
    def up(index):
        return (index+1)//2-1
    def left(index):
        return 2*index + 1
    def right(index):
        return 2*index + 2
```

Priority Queues

- Insert at the end of the array
  - but note the index

```
def insert(self, value):
    n = len(self.array)
    self.array.append(value)
    while n>0:
        parent = PQ.up(n)
        print(n, parent, 'indices')
        if self.array[parent] < value:
            self.array[n], self.array[parent] =
               self.array[parent], self.array[n]
            n = parent
        else:
            return
```

Priority Queues

- Adjust by swapping with parent
  - Index of current element is *n*

```
def insert(self, value):
    n = len(self.array)
    self.array.append(value)
    while n>0:
        parent = PQ.up(n)
        print(n, parent, 'indices')
        if self.array[parent] < value:
            self.array[n], self.array[parent] =
               self.array[parent], self.array[n]
            n = parent
        else:
            return
```

Priority Queues

- Calculate the parent node

```
def insert(self, value):
    n = len(self.array)
    self.array.append(value)
    while n>0:
        parent = PQ.up(n)

        if self.array[parent] < value:
            self.array[n], self.array[parent] =
               self.array[parent], self.array[n]
            n = parent
        else:
            return
```

Priority Queues

- And swap if necessary

```
def insert(self, value):
    n = len(self.array)
    self.array.append(value)
    while n>0:
        parent = PQ.up(n)

        if self.array[parent] < value:
            self.array[n], self.array[parent] =
               self.array[parent], self.array[n]
            n = parent
        else:
            return
```

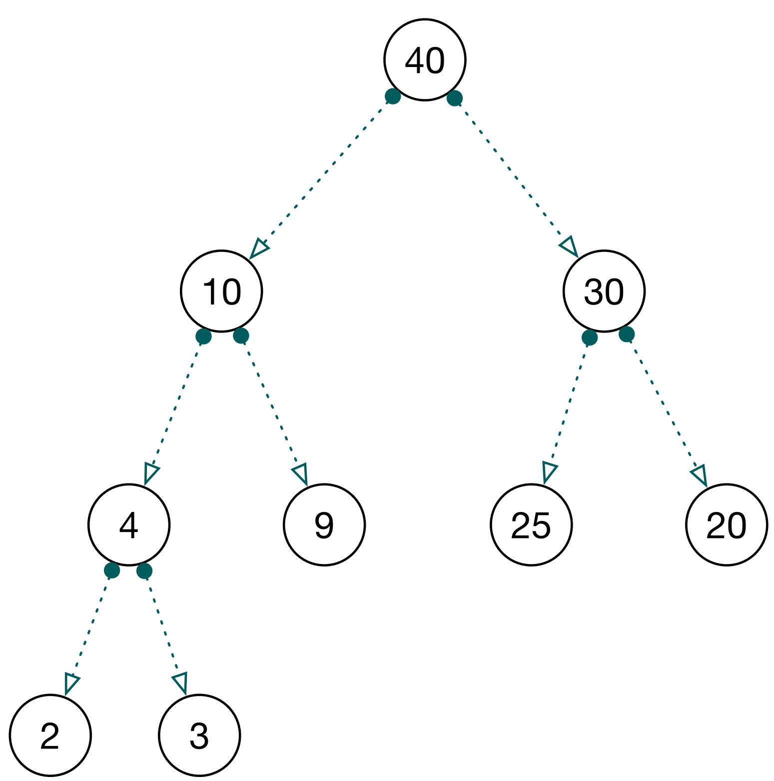
Priority Queues

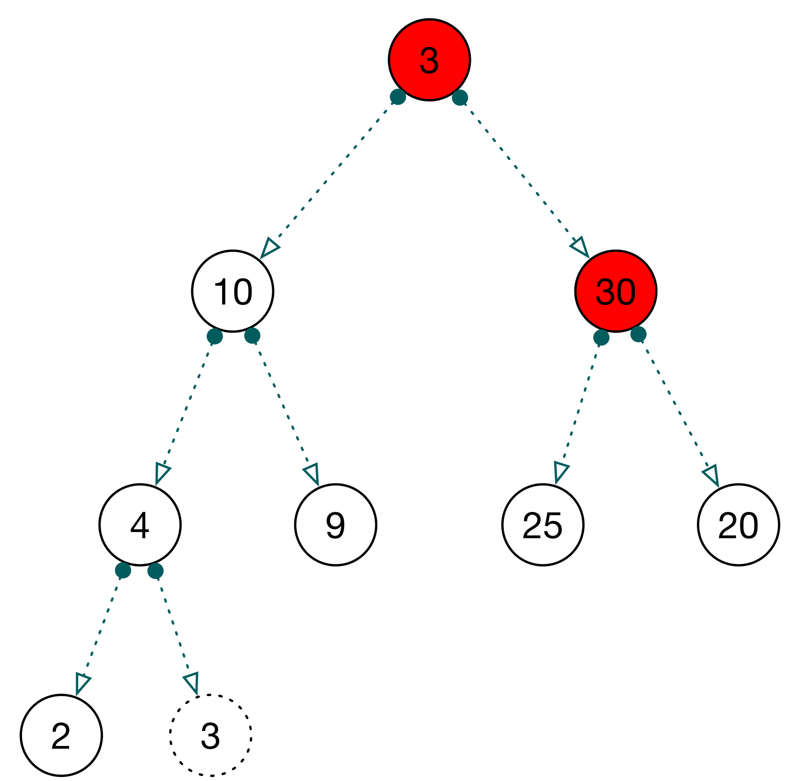
- Then reset the index

```
def insert(self, value):
    n = len(self.array)
    self.array.append(value)
    while n>0:
        parent = PQ.up(n)
        print(n, parent, 'indices')
        if self.array[parent] < value:
            self.array[n], self.array[parent] =
               self.array[parent], self.array[n]
            n = parent
        else:
            return
```

Priority Queues

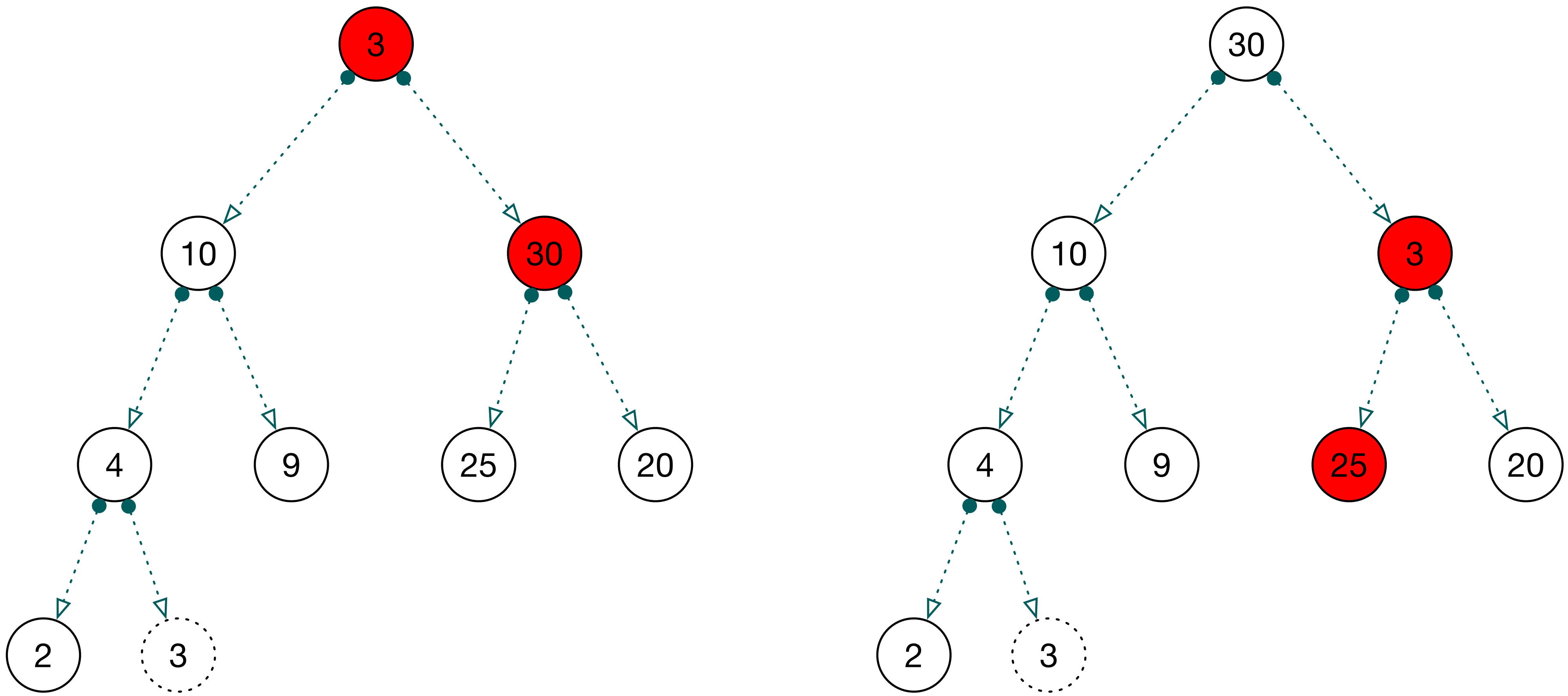
- Extract maximum:
  - Maximum is always at position 0
  - Swap its value with the last element in the array
  - Then heapify:



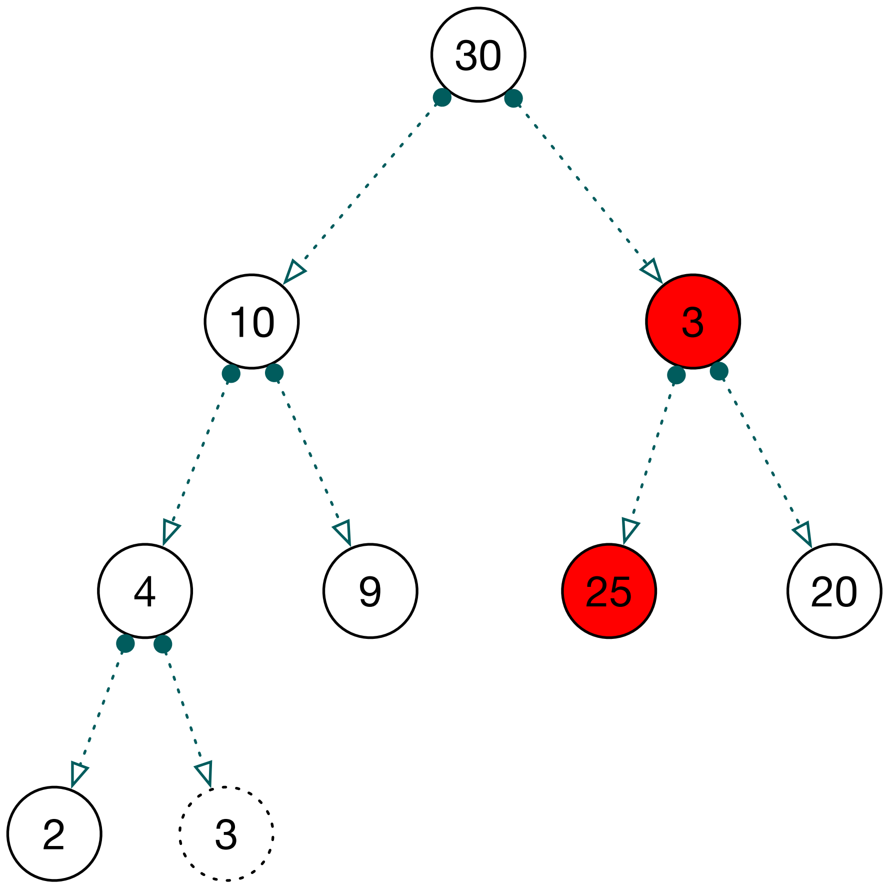


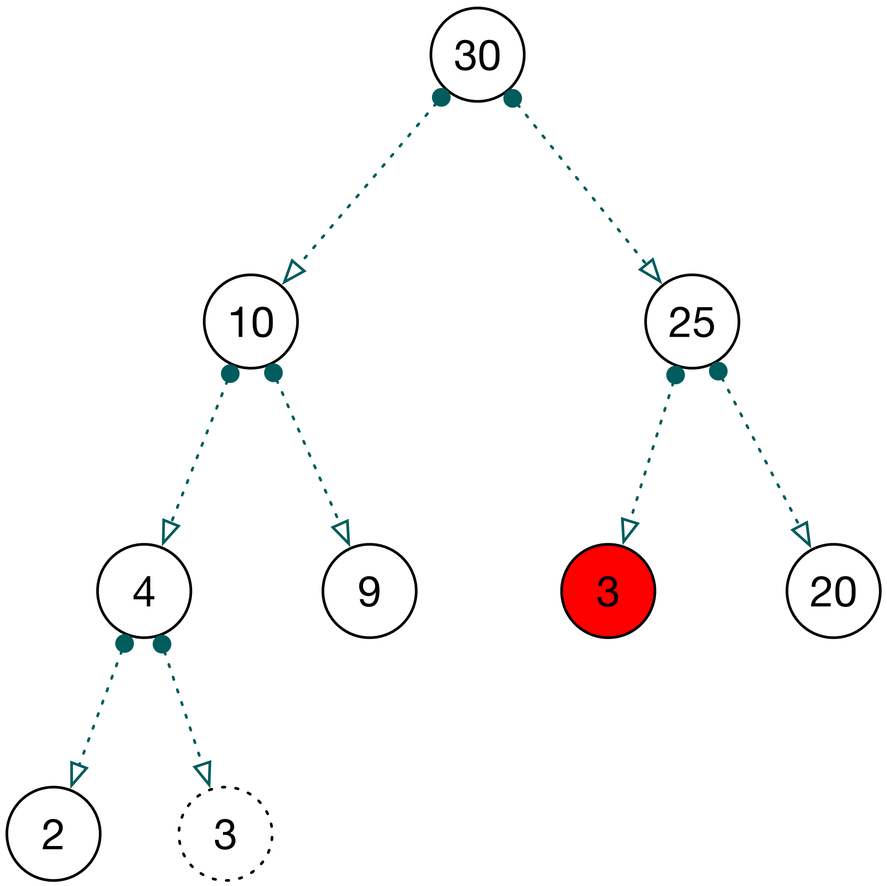
Priority Queues

- This is also recursive, but proceeds from top to bottom



Priority Queues





Priority Queues

- Swap last and first node
- Delete from node

```
def get_max(self):
     ret_val = self.array[0]
     last = self.array[-1]
     del self.array[-1]
     self.array[0] = last
     n=0
```

Priority Queues

- Now recursively recover the heap property
  - Make case distinctions according to whether
    - both children exist
    - only the left child exist
    - no children present

Priority Queues

- Both children exist

```
 def get_max(self):
     ...
     while n < len(self.array):
          left = PQ.left(n)
          right = PQ.right(n)
          if right < len(self.array):
              if self.array[n] > self.array[left] and
                        self.array[n] > self.array[right]:
                  return ret_val
              if self.array[left] < self.array[right]:
                  m = right
              else:
                  m = left
              self.array[n], self.array[m] = self.array[m],
self.array[n]
              n = m
```

Priority Queues

- Heap property is not violated

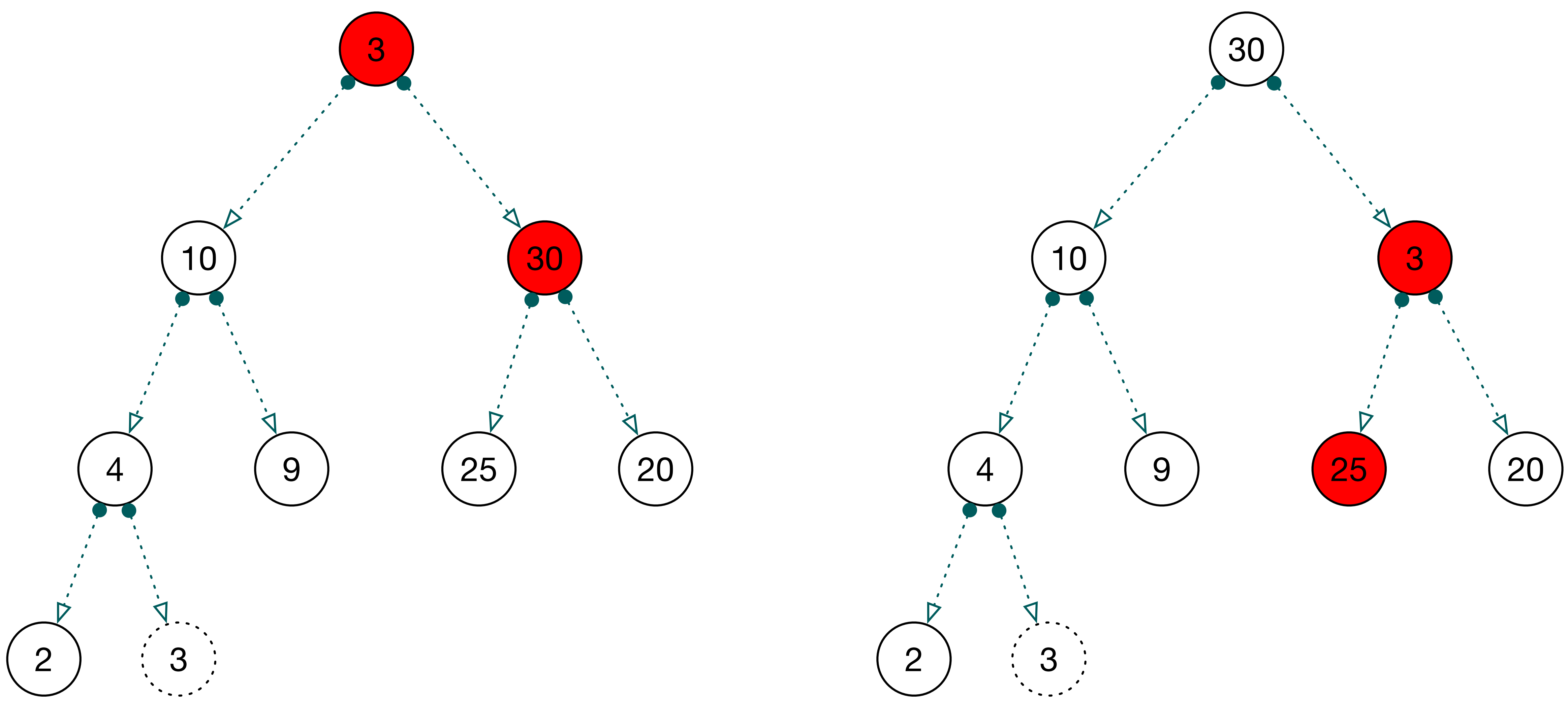
```
def get_max(self):
    ...
    while n < len(self.array):
        left = PQ.left(n)
        right = PQ.right(n)
        if right < len(self.array):
            if self.array[n] > self.array[left] and
                    self.array[n] > self.array[right]:
                return ret_val
            if self.array[left] < self.array[right]:
                m = right
            else:
                m = left
            self.array[n], self.array[m] = self.array[m],
self.array[n]
            n = m
```

Priority Queues

- Select the larger of the two children for swapping

```
def get_max(self):
    ...
    while n < len(self.array):
        left = PQ.left(n)
        right = PQ.right(n)
        if right < len(self.array):
            if self.array[n] > self.array[left] and
                      self.array[n] > self.array[right]:
                return ret_val
            if self.array[left] < self.array[right]:
                m = right
            else:
                m = left
            self.array[n], self.array[m] =
                  self.array[m], self.array[n]
            n = m
```

Priority Queues



Priority Queues

- Swap

```
def get_max(self):
    ...
    while n < len(self.array):
        left = PQ.left(n)
        right = PQ.right(n)
        if right < len(self.array):
            if self.array[n] > self.array[left] and
                      self.array[n] > self.array[right]:
                return ret_val
            if self.array[left] < self.array[right]:
                m = right
            else:
                m = left
            self.array[n], self.array[m] =
                  self.array[m], self.array[n]
            n = m
```

Priority Queues

- Swap

```
def get_max(self):
   ...
   while n < len(self.array):
        left = PQ.left(n)
        right = PQ.right(n)
        if right < len(self.array):
            if self.array[n] > self.array[left] and
                      self.array[n] > self.array[right]:
                return ret_val
            if self.array[left] < self.array[right]:
                m = right
            else:
                m = left
            self.array[n], self.array[m] =
                  self.array[m], self.array[n]
            n = m
```

Priority Queues

- And do not forget to set yourself up for recursion

```
def get_max(self):
   ...
   while n < len(self.array):
        left = PQ.left(n)
        right = PQ.right(n)
        if right < len(self.array):
            if self.array[n] > self.array[left] and
                      self.array[n] > self.array[right]:
                return ret_val
            if self.array[left] < self.array[right]:
                m = right
            else:
                m = left
            self.array[n], self.array[m] =
                  self.array[m], self.array[n]
            n = m
```

Priority Queues

- Only one child can exist (but then it has to be the left one)
    - Heap property might not be violated

```
elif left < len(self.array):
    if self.array[n] > self.array[left]:
        return ret_val
    m = left
    self.array[n], self.array[m] =
              self.array[m], self.array[n]
    n = m
```

Priority Queues

- Only one child can exist (but then it has to be the left one)
  - But if it is, we have only one candidate for swapping

```
elif left < len(self.array):
     if self.array[n] > self.array[left]:
         return ret_val
     m = left
     self.array[n], self.array[m] =
               self.array[m], self.array[n]
     n = m
```

Priority Queues

- Per defensive programming, we pretend that we might have to go on:

```
elif left < len(self.array):
     if self.array[n] > self.array[left]:
          return ret_val
     m = left
     self.array[n], self.array[m] =
               self.array[m], self.array[n]
     n = m
```

Priority Queues

- Difficult Homework:
  - Extract Maximum and insertion of a new element are sometimes combined
  - In this case, we can save work by:
    - inserting the new element at the beginning of the array
    - work ourselves downwards to restore the heap property
  - Implement this

Priority Queues

- Other operations:
  - peek
    - returns the maximum, but does not remove it
  - is_empty
    - checks whether the array is empty

Priority Queues

- Costs of operations
  - Priority queue with n elements uses $\log_2(n)$ steps in order to heapify
  - Peek and is_empty run in constant time

Priority Queues

- Python implementation of priority queues
  - heapq implements a minimum heap
  - Uses a Python list

```
heapq.heappush(lista, element)

heapq.heappop(lista)
```

Priority Queues

- This is an efficient implementation
    - We can "kludge" a max heap implementation for integers by observing that the maximum of numbers is the negative of the negative integers

```
def smallpush(lista, element):
    heapq.heappush(lista, -element)
def smallpop(lista):
    return -heapq.heappop(lista)
```

Running Medians

- Task:
  - We are given a stream of numbers
    - At any time, want to be able to determine the median of these numbers
- Example:
  - We get 5, 3, 1, 10, 2
  - Median is now 3
  - We then get 12, 1, 2
    - We have seen 1,1,2,2,3,5,10,12
  - Median is now 2.5 (mean of 2 and 3)

Running Medians

- Naïve implementation
  - Just keep an ordered list around
- Better way:
  - Keep two sublists of equal size
    - Small and Big
    - All elements in Small are smaller than all elements in Big
    - Use heaps in order to easily extract the maximum of Small and the minimum of Big

Running Medians

- Adding a new number:
  - If the left heap is smaller, then insert there
  - If the left and right heap have equal size, insert in the right heap
    - But need to maintain the invariant:
      - All elements in the left heap are smaller (or equal) than all elements in the right heap

Running Medians

- Example: Inserting 5 into
  - Left: 0, 1, 1, 2, 2 Right: 3, 4, 6, 7, 7, 9
- We need to insert into Left, but this violates the invariant
  - Extract the minimum from right (3)
  - Add the minimum to the left
  - Add 5 to right
  - Left: 0, 1, 1, 2, 2, 3 Right: 4, 5, 6, 7, 7, 9

Running Medians

- Insert another 5:
  - Left: 0, 1, 1, 2, 2, 3   Right: 4, 5, 6, 7, 7, 9
- Rule say insert to the Right:
  - Since max(left) < 5:
    - No problem:
  - Left: 0, 1, 1, 2, 2, 3   Right: 4, 5, 5, 6, 7, 7, 9

Running Medians

- Insert another 5:
  - Insert into Left:
    - But min(right) = 4 which is smaller than 5
  - Inserting 5 into left violates the invariant
    - Need to do something about it:
      - Extract minimum from Right
      - Insert this minimum into Left
      - Insert new element into Right
  - Left: 0, 1, 1, 2, 2, 3, 4   Right: 5, 5, 6, 7, 7, 9

Running Medians

- Calculating medians:
  - If len(Left) < len(Right):
    - Median is peek(Right)
  - Otherwise:
    - Median is (peek(Right)+peek(Left))/2

Tiered Bitvectors

- Given a finite universe indexed by $\{0,1,\ldots,n-1\}$:
  - $U = \{x_i : i \in \{0,1,\ldots,n-1\}\}$
- A bit vector represents subsets of U
  - If $\delta_{i,S} = \begin{cases} 0 & \text{if } i \in S \\ 1 & \text{if } i \notin S \end{cases}$ (Kronecker delta)
  - $S \subset U$ corresponds to the bit-vector
    - $(\delta_{i,S} : i \in \{0,1,\ldots,n-1\})$

Tiered Bitvectors

- To insert x_i into the set:
  - Set bit i to 1
- To delete x_i from the set:
  - Set bit i to 0
- To lookup whether $x_i \in S$:
  - Check value of bit i

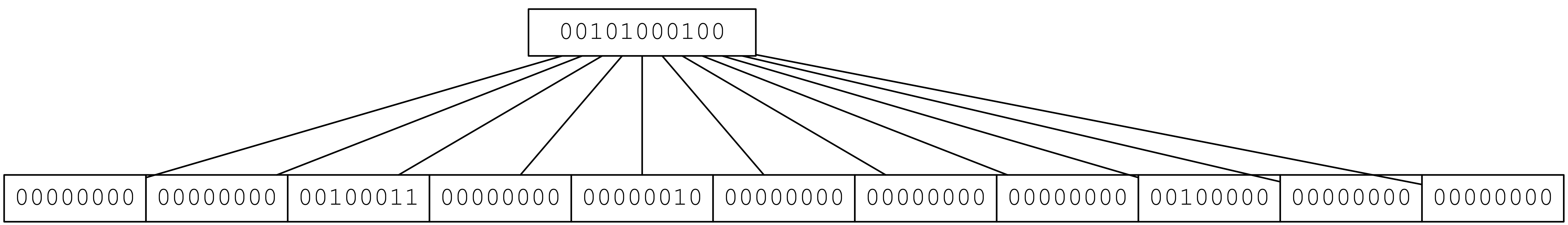
Tiered Bitvectors

- MinIndex, MaxIndex, PredecessorIndex, SuccessorIndex are extremely slow
  - $\Theta(|U|)$

Tiered Bitvectors

- If memory is (as is typical) accessed via cache-lines
  - Words of length $L = 64$B
- Break the universe into $|U|/L$ pieces of size L
  - $U_0, U_1, U_2, \ldots, U_{n/L}$
- Introduce a master bitvector by
  - $\Delta_{i,S} = \begin{cases} 0 & \text{if } S \cap U_i = \emptyset \\ 1 & \text{if } S \cap U_i \neq \emptyset \end{cases}$

Tiered Bitvectors



Tiered Bitvectors

- Ordered dictionary operations run in time
  - $O(U/L + L)$
- This is minimized when $L = \sqrt{U}$

Tiered Bitvectors

- What are the operations?

Tiered Bitvectors

- Inserting to a set:
  - Set master bitvector bit
  - Set partial bitvector bit
- Deleting from a set
  - Reset partial bitvector bit
  - If partial bitvector is empty: Reset master bitvector bit

Tiered Bitvectors

- is-empty:
  - Check master bitvector only has entries 0
- min:
  - One min operation on the master bitvector, one min operation on partial bitvector

Tiered Bitvectors

- Obviously, we can extend this from two tiers to many tiers.
  - Result is a tree
  -

Fibonacci Heaps

Mergeable Heaps

- Mergeable Heaps are ADS with
  - Make-Heap
  - Insert(*H*, *x*)
  - Minimum(*H*)
  - Extract-Min(*H*)
  - Union(H_1, H_2)
- Fibonacci heaps in addition have
  - Decrease-Key(*H, x, new_val*)
  - Delete(*H,x*)

Mergeable Heaps

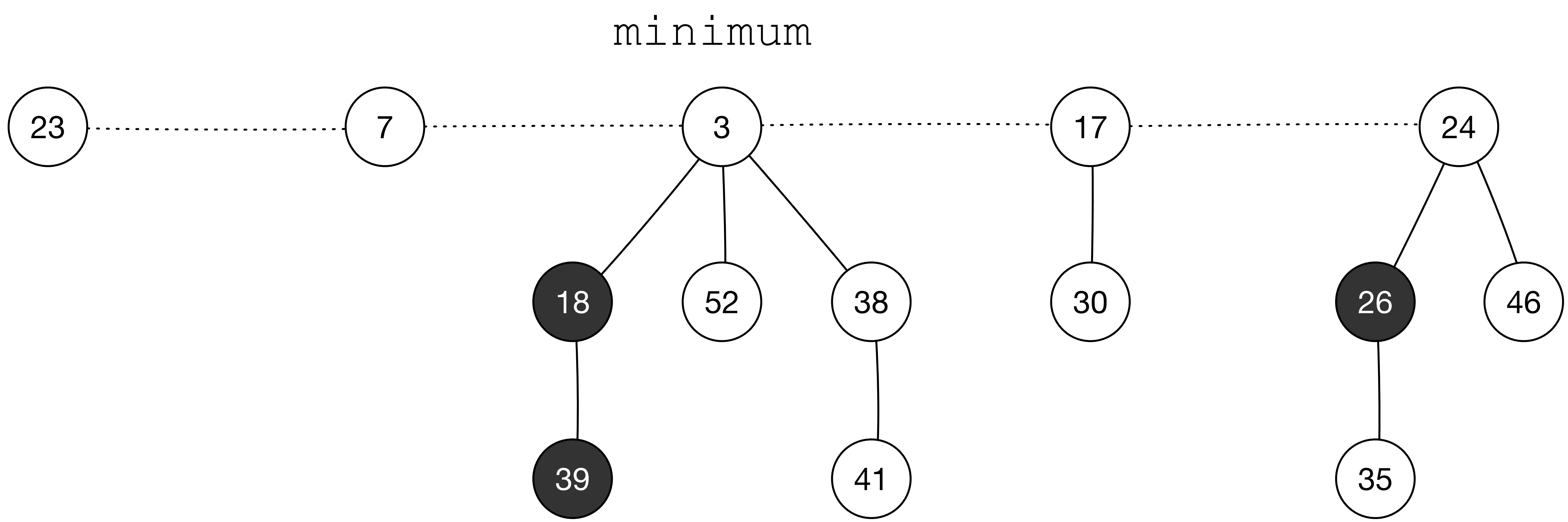
Procedure	Binary Heap (worst-case)	Fibonacci Heap (amortized)
Make-Heap	$\Theta(1)$	$\Theta(1)$
Insert	$\Theta(\log n)$	$\Theta(1)$
Minimum	$\Theta(1)$	$\Theta(1)$
Extract-Min	$\Theta(\log n)$	$O(\log n)$
Union	$\Theta(n)$	$\Theta(1)$
Decrease-key	$\Theta(\log n)$	$\Theta(1)$
Delete	$\Theta(\log n)$	$O(\log n)$

Fibonacci Heaps

- Fibonacci heaps:
  - Useful when Extract-Min and Delete are rare
  - E.g. graph algorithms where we use decrease-key in order to update edges
    - Minimum spanning trees
    - Single-source shortest paths
  -

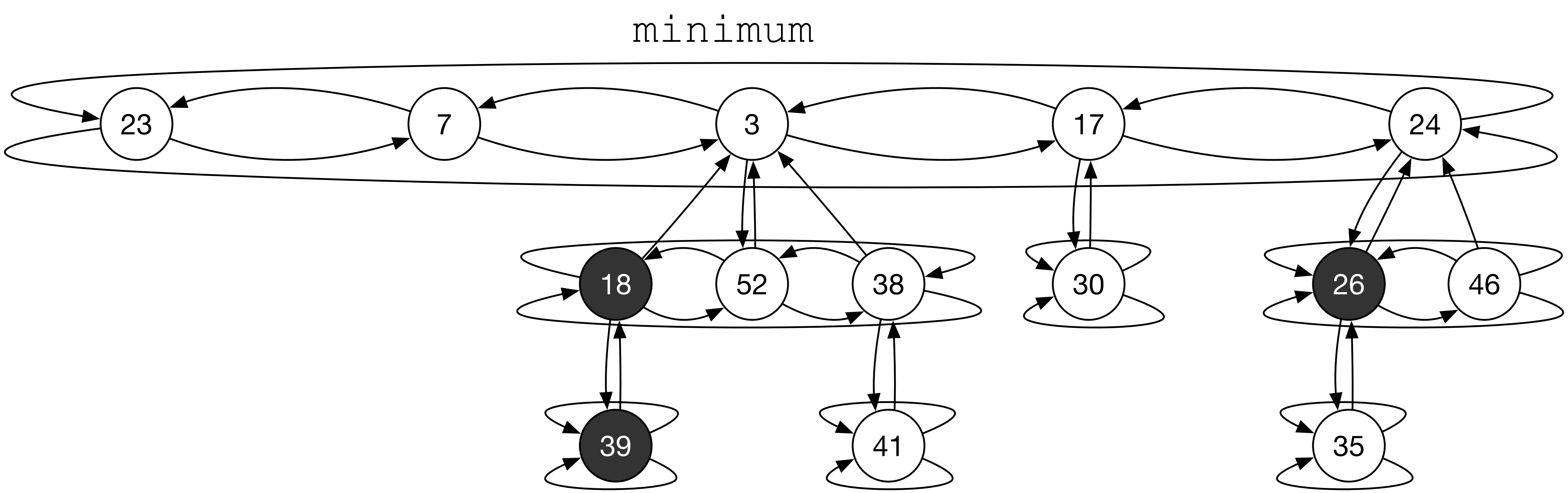
Fibonacci Heaps

- Fibonacci heap:
  - Collection of rooted trees that are min-heap ordered:
    - Every child's key is larger than its parent's key
- Example:



Fibonacci Heaps

- Each node has a link to their parent
- All children (including root list) are in a double linked child list
- Parent has one link to the child list



Fibonacci Heaps

- Double linked list:
  - Allows constant time inserts and fusion

Fibonacci Heaps

- Each node has attributes
  - Number of children in degree
  - Node x has mark x.mark to indicate whether node x has lost a child since the last time x was made a child of another node
    - Newly created nodes are unmarked
    - Node x becomes unmarked whenever it is made child of another node
    - Used for DecreaseKey operation

Fibonacci Heaps

- Access to a Fibonacci node through pointer to minimum key node
- Each Fibonacci node stores
  - Links to left and right sibling
  - Link to parent unless root
  - Link into the child list

Fibonacci Heaps

- Use potential method
  - $t(H)$: number of trees in the root list
  - $m(H)$: number of marked nodes
- Potential $\Phi(H) = t(H) + 2m(H)$
  - Unit of potential can pay for all constant time operations

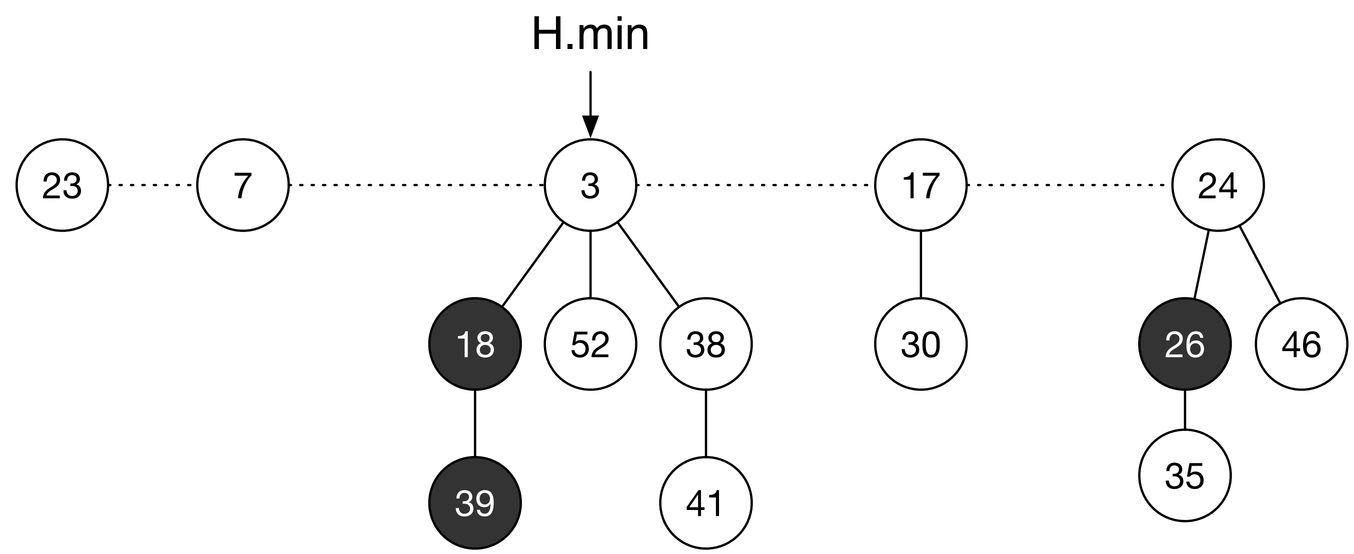
Fibonacci Heap Operations

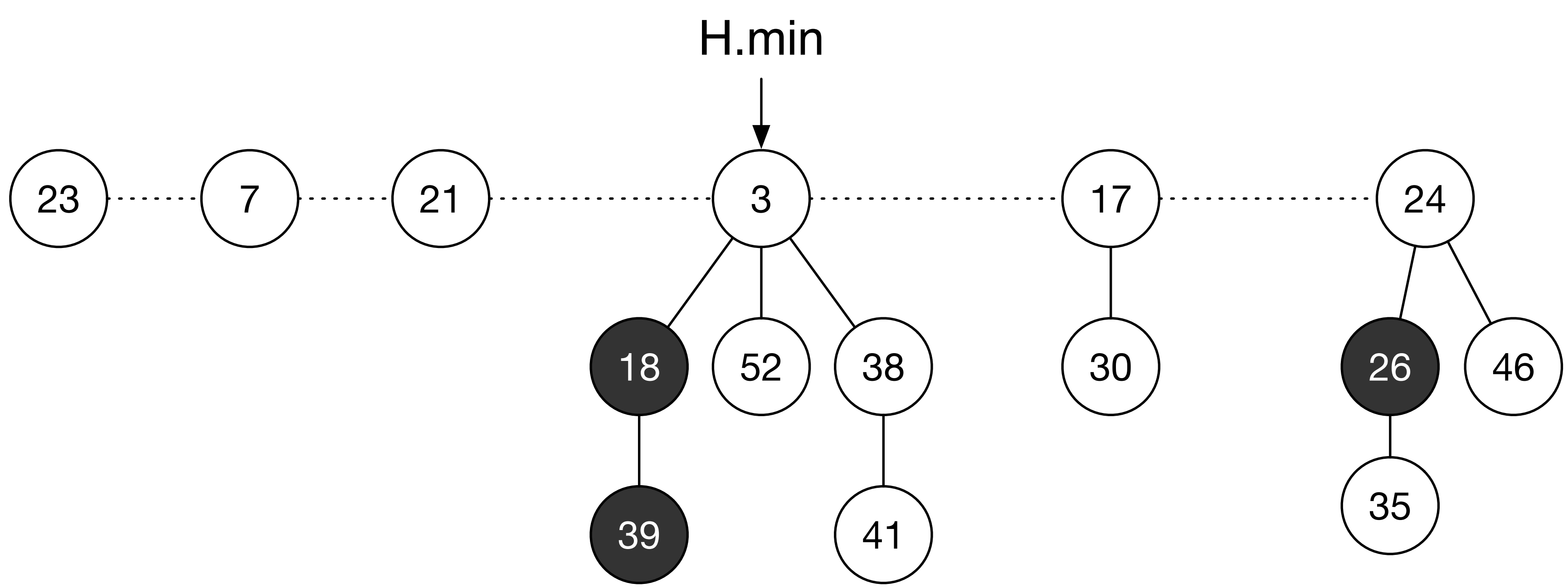
- Creating a new Fibonacci Heap
  - $H.n = 0$
  - $H.\min =$ Null
  - $\Phi(H) = 0$
  - amortized cost is $O(1)$

Fibonacci Heaps

- Insert(H,x)
  - Create an otherwise empty tree with x as root
  - If there is no element in the heap (H.min = Null):
    - Create a root list for H containing x only
  - Otherwise
    - Insert the new tree into the root list
    - Check whether H.min needs to be updated
- Amortized costs: $O(1) + 1 = O(1)$

Fibonacci Heaps





Fibonacci Heaps

- Minimum
    - Just returns a pointer to the minimum node
- Amortized costs is $O(1)$

Fibonacci Heaps

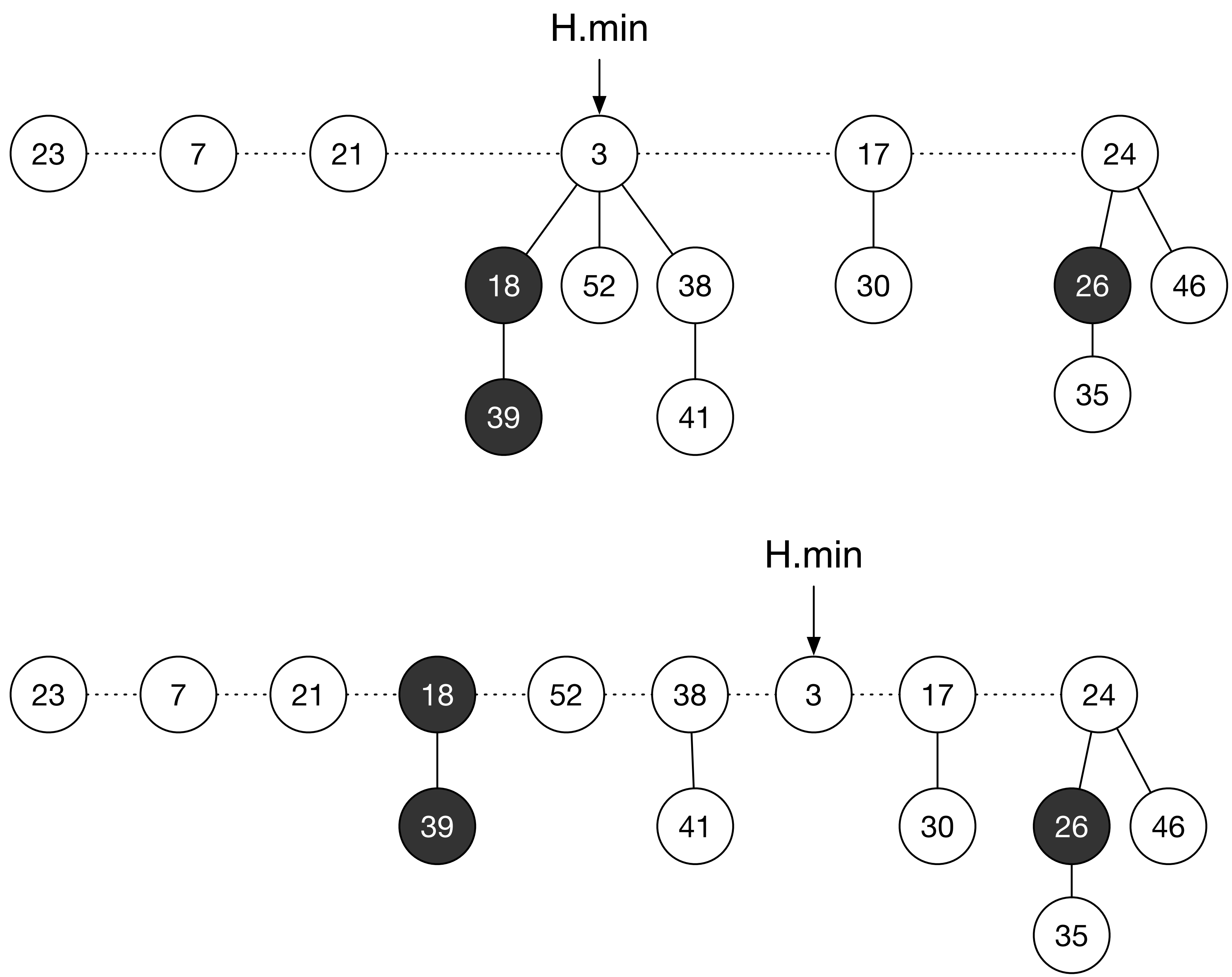
- Uniting two heaps
  - Concatenate the root lists
- Change in potential is zero
- Amortized costs is $O(1)$

Fibonacci Heaps

- Extracting the minimum
  - This is where we consolidate
  -

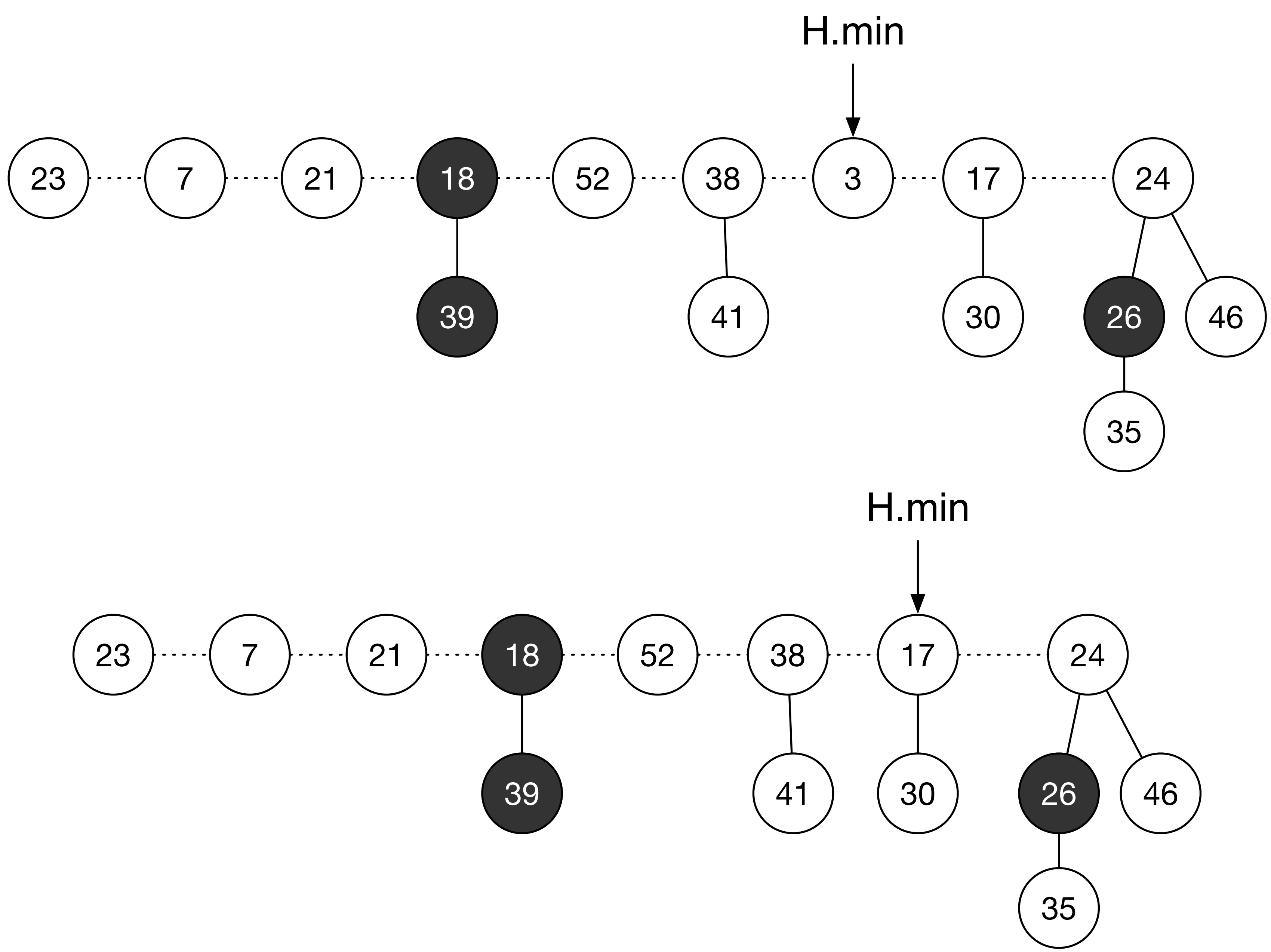
```
z=H.min
if z ≠ NULL:
   for each child x of z:
       add x to the root list of H
       x.p = NULL
    remove z from root list
    if z == z.right: #z only node
        H.min = NULL
    else:
        H.min = z.right
        CONSOLIDATE(H)
    H.n -= 1
return z
```

Fibonacci Heaps



Moving children of H.min into the root list

Fibonacci Heaps



Removing and returning H.min

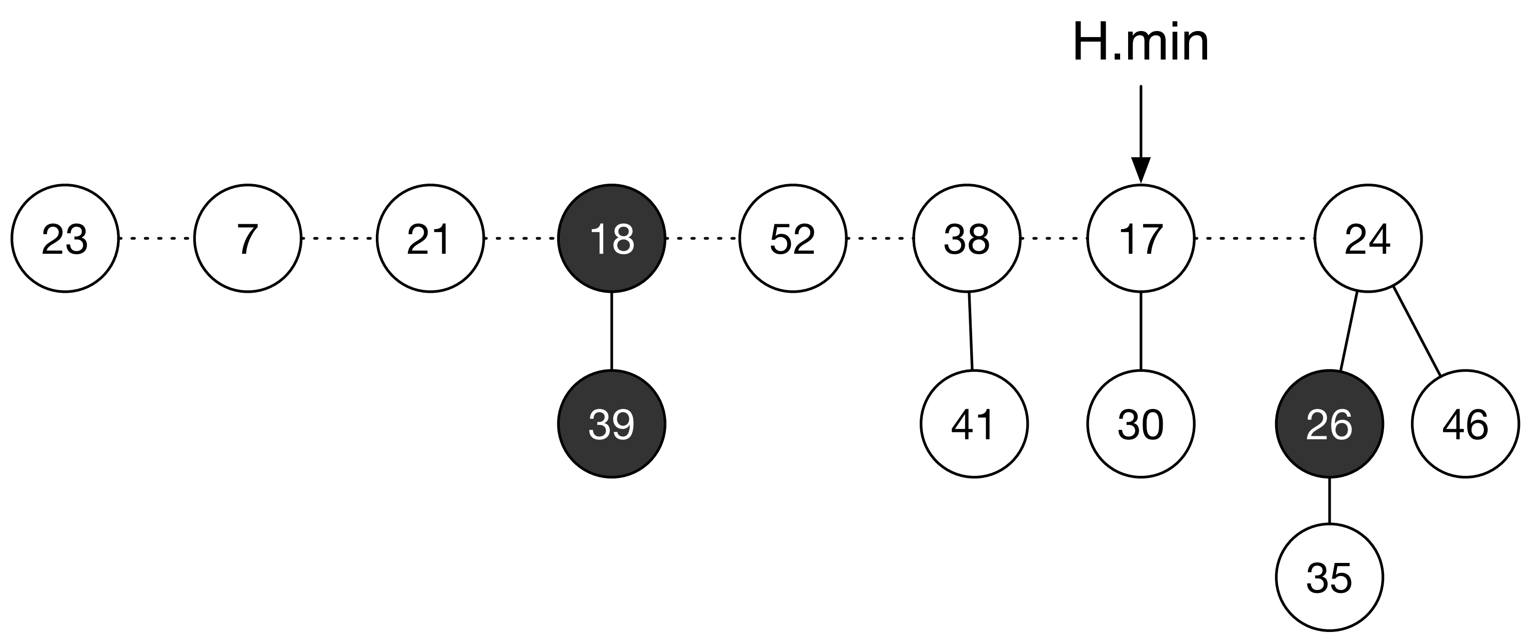
Fibonacci Heaps

- Consolidation:
  - Repeat:
    - Find two nodes in the root list with trees of the same height
      - Starting at the current link H.min
    - Unify the two trees making the smaller one the root
      - Clears mark on the looser

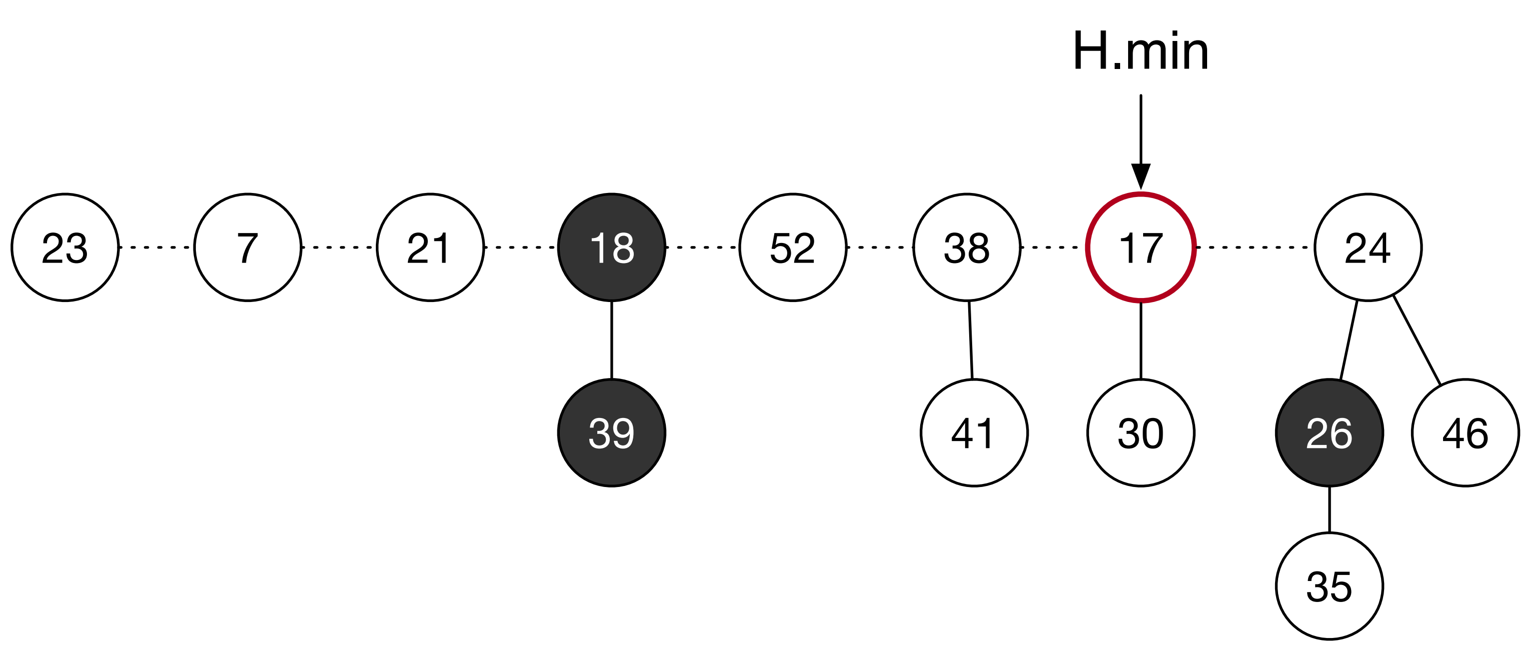
Fibonacci Heaps

- Consolidation:
  - To find nodes in the root list that can be merged
    - Create an array A[0 … D(H.n)] of nodes in the root tree
      - D(H.n) is the maximum degree of a tree rooted in a node in the root list
    - Fill into A
    -

Fibonacci Heaps

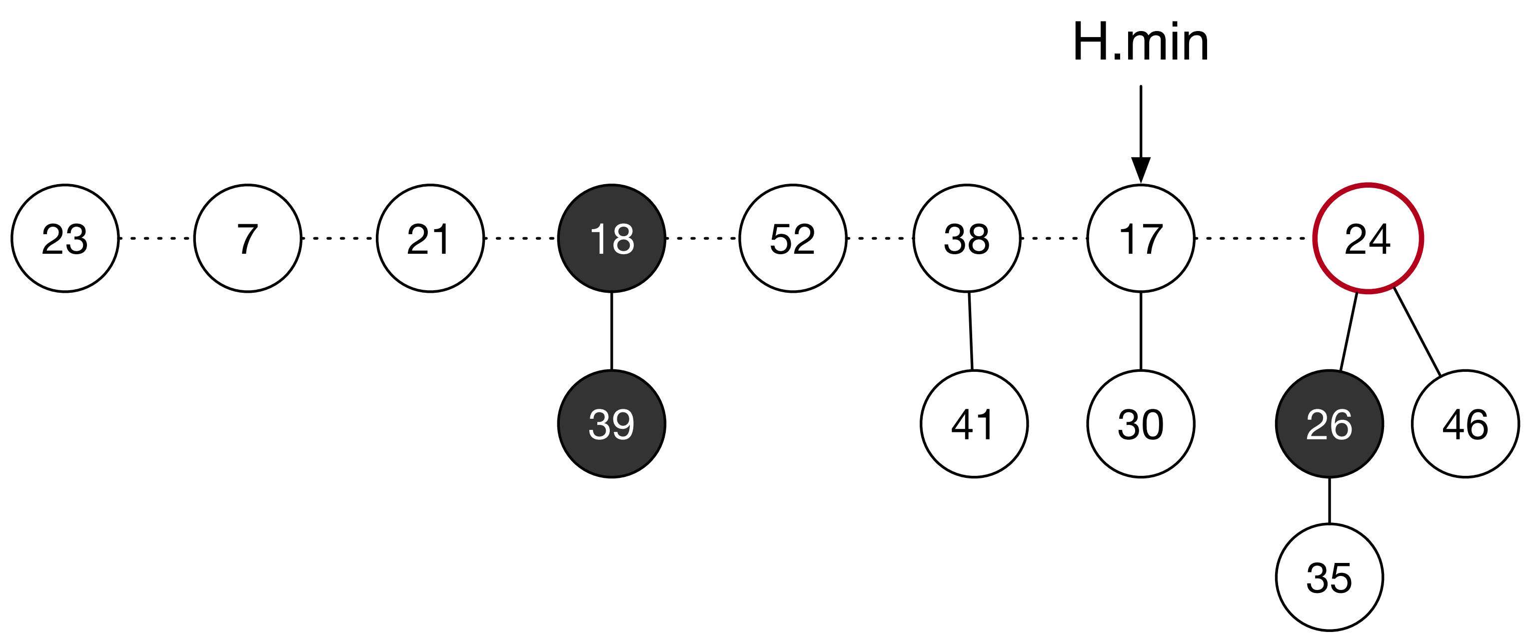


A[0]: NULL
A[1]: NULL
A[2]: NULL
A[3]: NULL



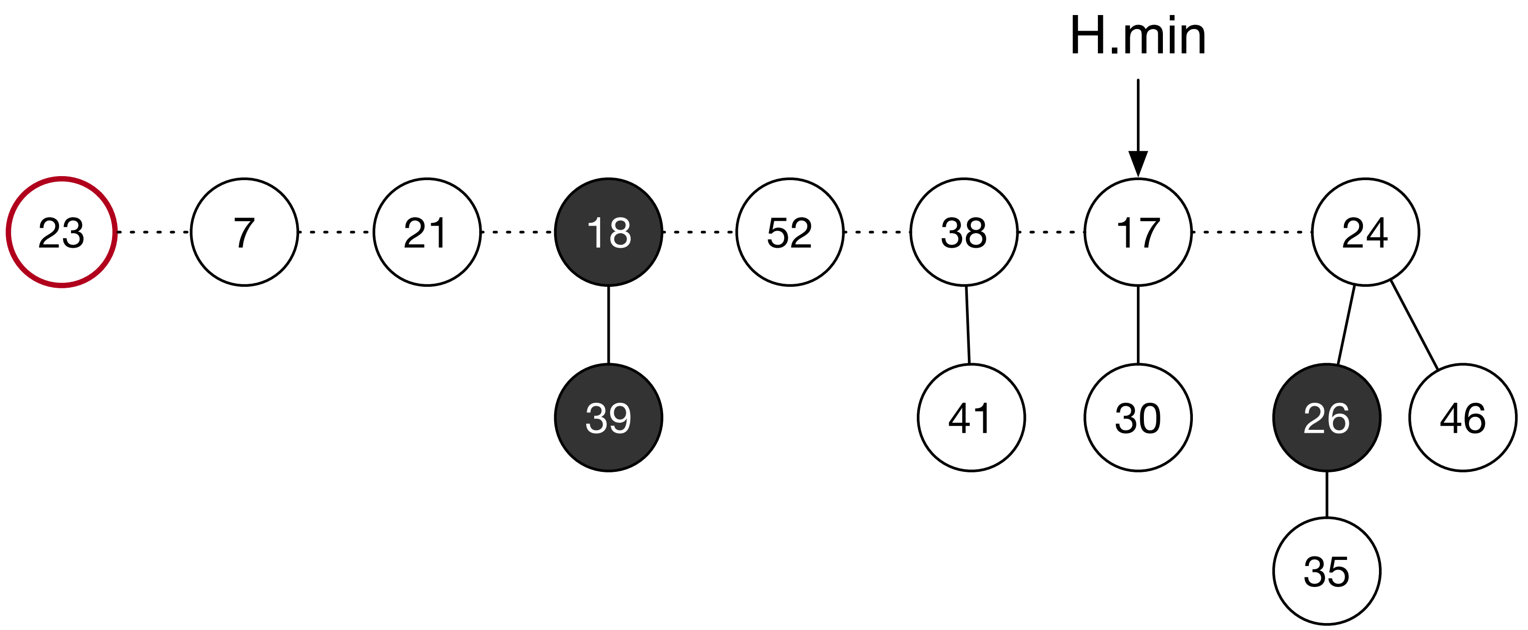
A[0]: NULL
A[1]: 17
A[2]: NULL
A[3]: NULL

Fibonacci Heaps



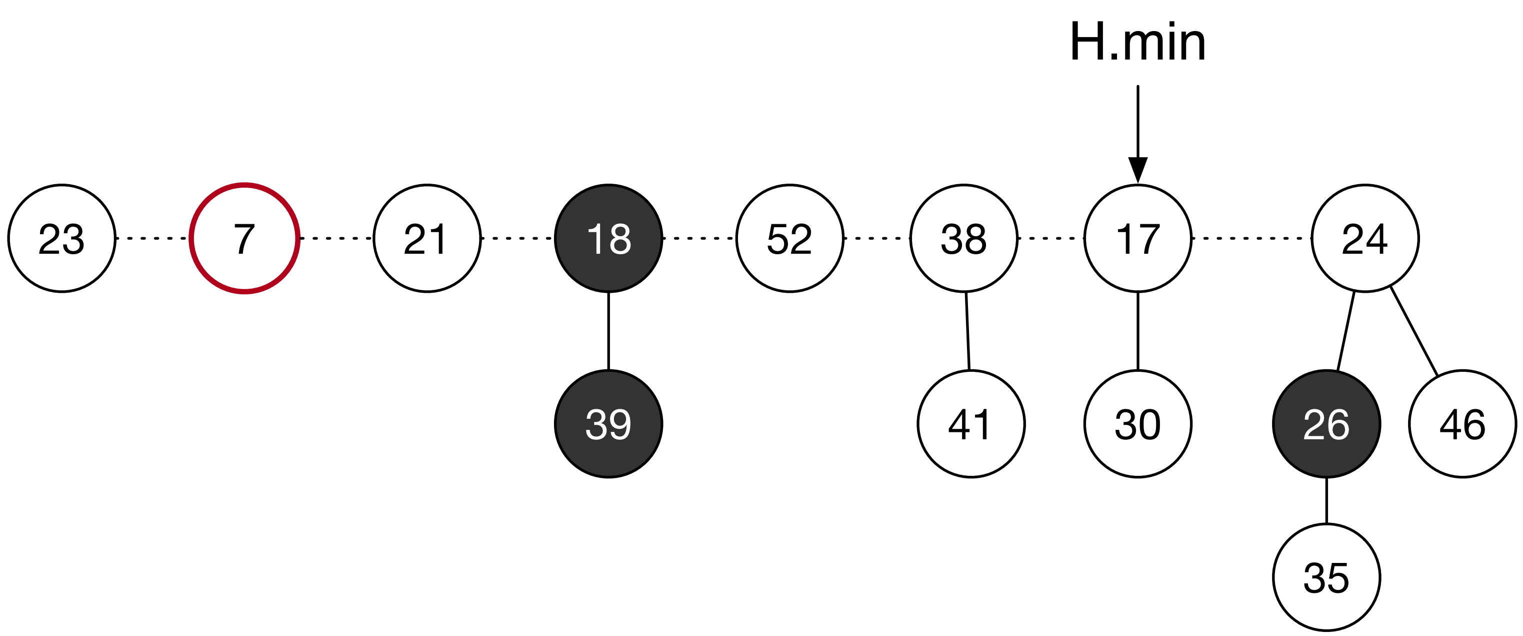
A[0]: NULL
A[1]: 17
A[2]: 24
A[3]: NULL

Fibonacci Heaps



A[0]: 23
A[1]: 17
A[2]: 24
A[3]: NULL

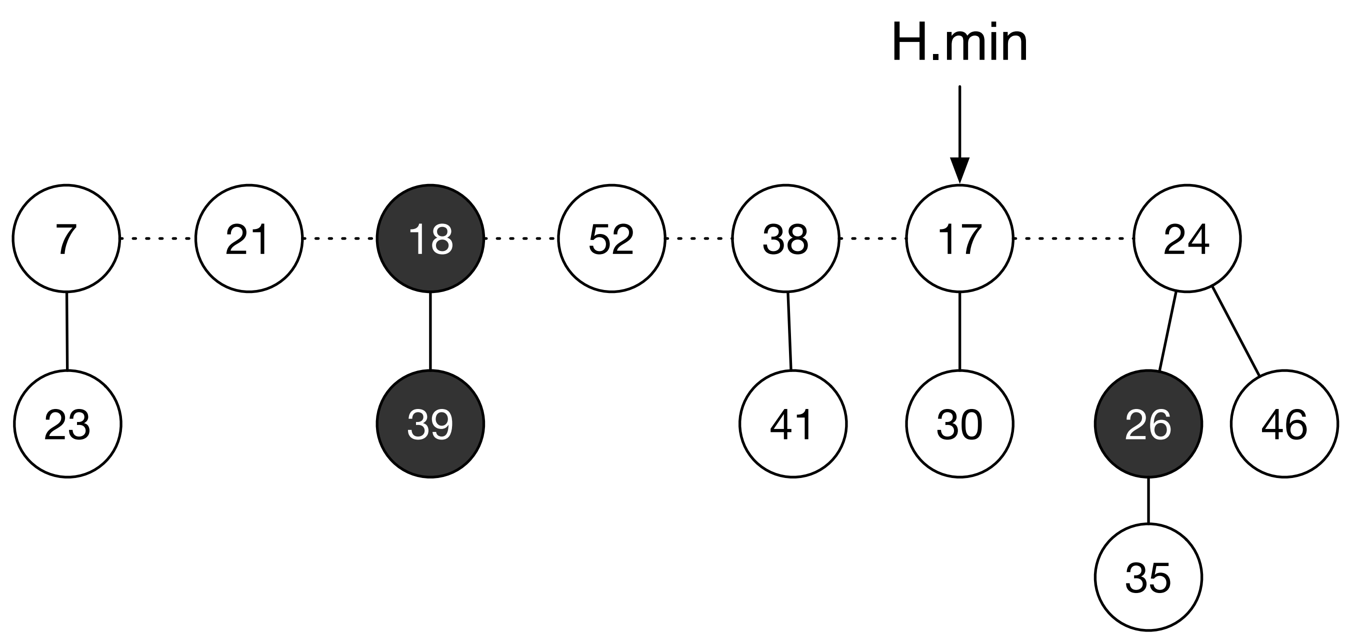
Fibonacci Heaps



A[0]: 23, 7
A[1]: 17
A[2]: 24
A[3]: NULL

We cannot insert 7 into A[0], so we combine

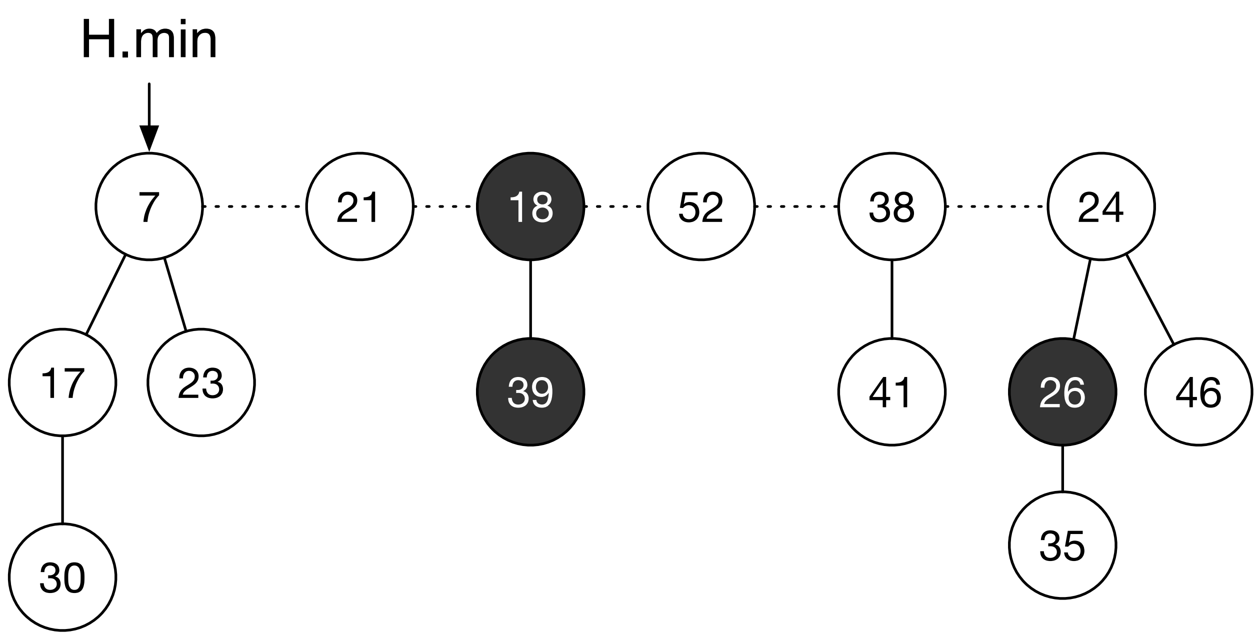
Fibonacci Heaps



A[0]: NULL
A[1]: 17, 7
A[2]: 24
A[3]: NULL

We remove 7 from the array. We then merge.
After merging, we try to insert 7 into A[1]. Since there is already an occupant there, we find another merge candidate

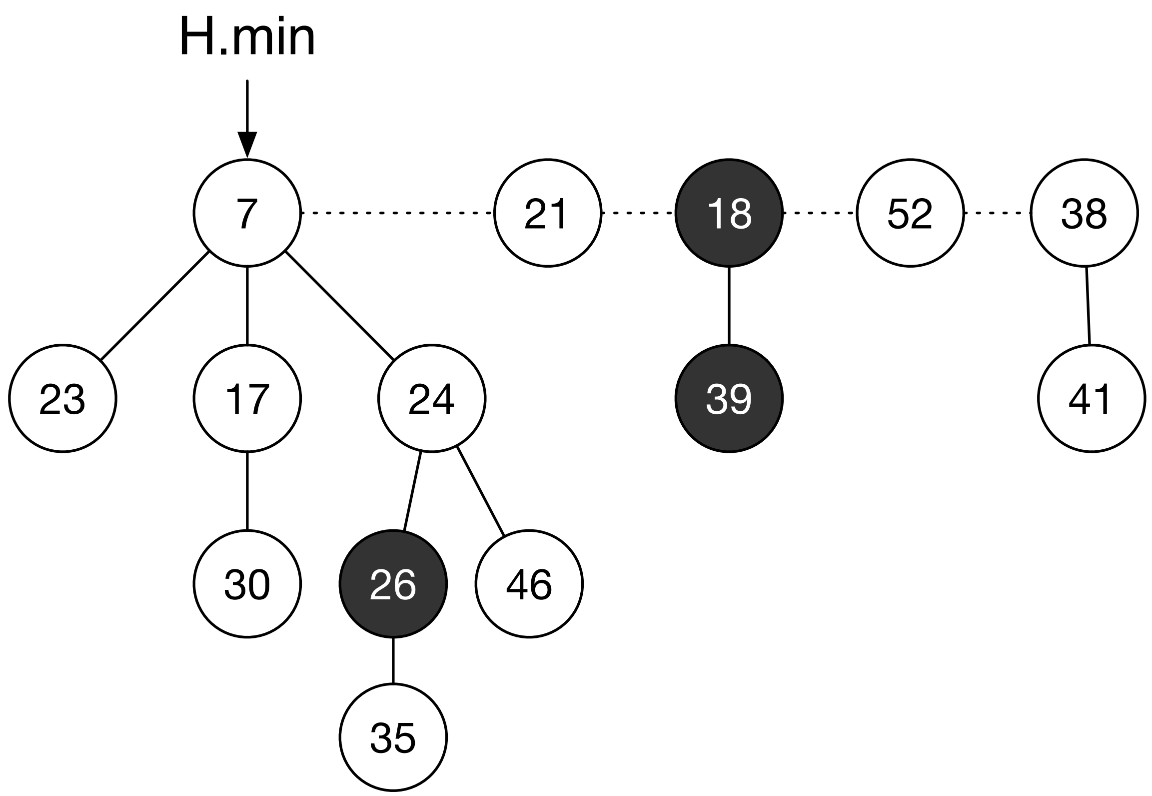
Fibonacci Heaps



A[0]: NULL
A[1]: NULL
A[2]: 24, 7
A[3]: NULL

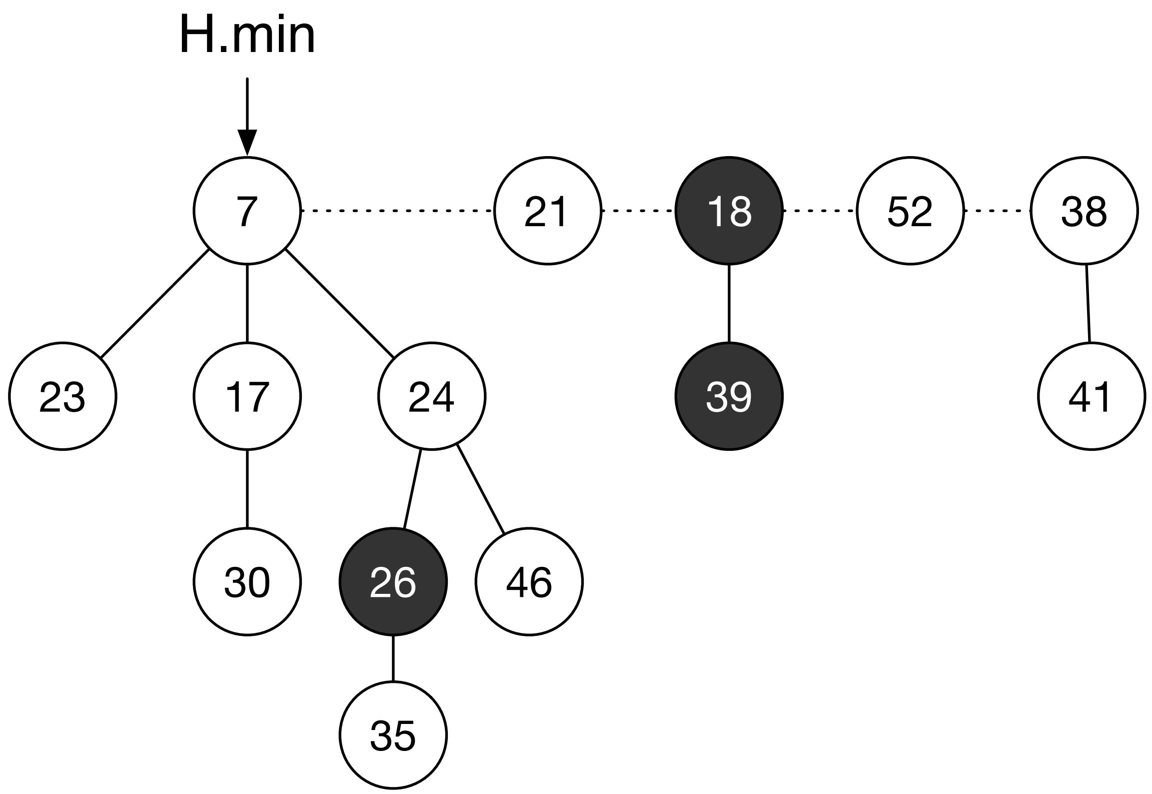
Inserting the new tree into A gives us another merge candidate

Fibonacci Heaps



A[0]: NULL
A[1]: NULL
A[2]: NULL
A[3]: 7

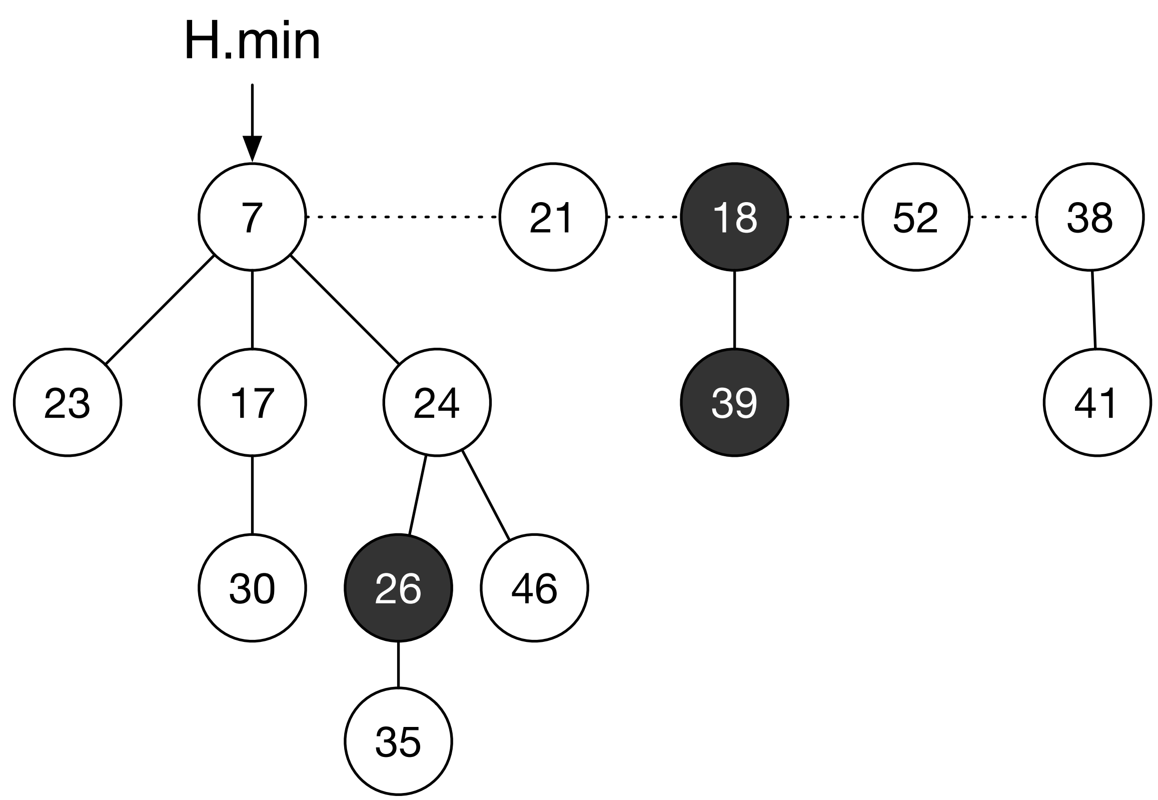
Fibonacci Heaps



A[0]: 21
A[1]: NULL
A[2]: NULL
A[3]: 7

We insert 21 into the array.

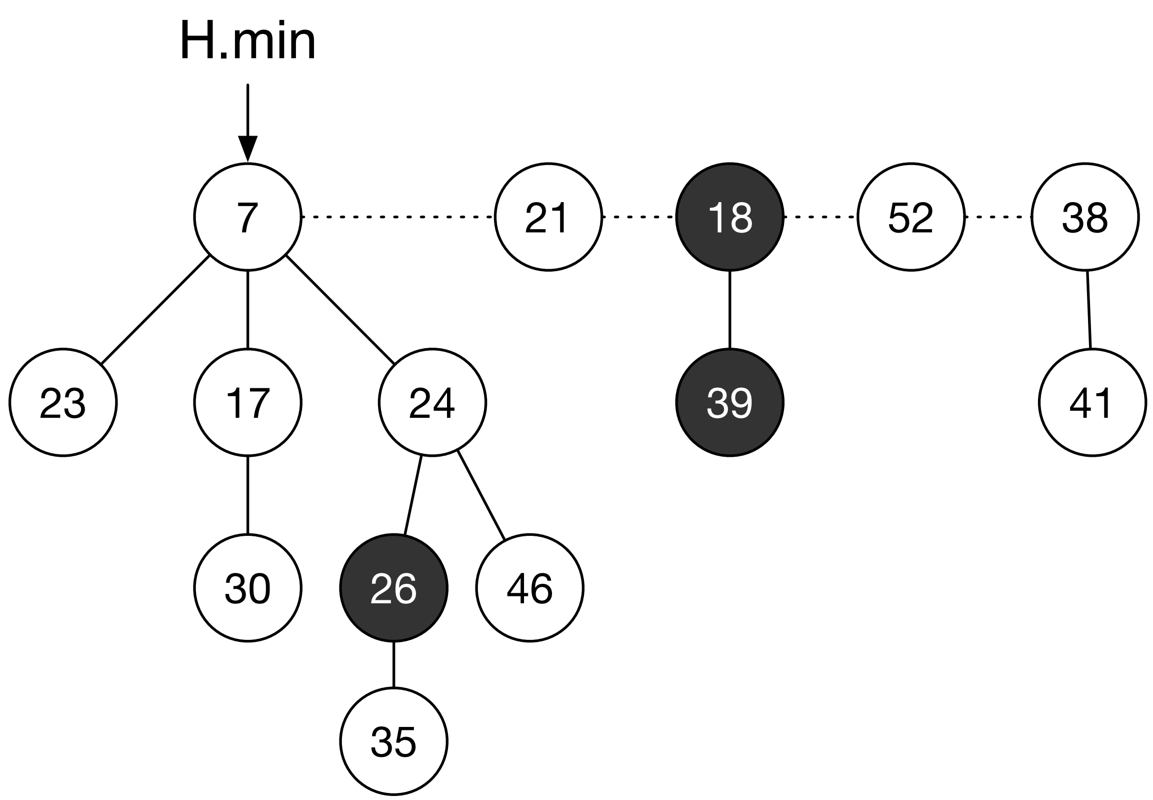
Fibonacci Heaps



A[0]: 21
A[1]: 18
A[2]: NULL
A[3]: 7

And then insert 18.

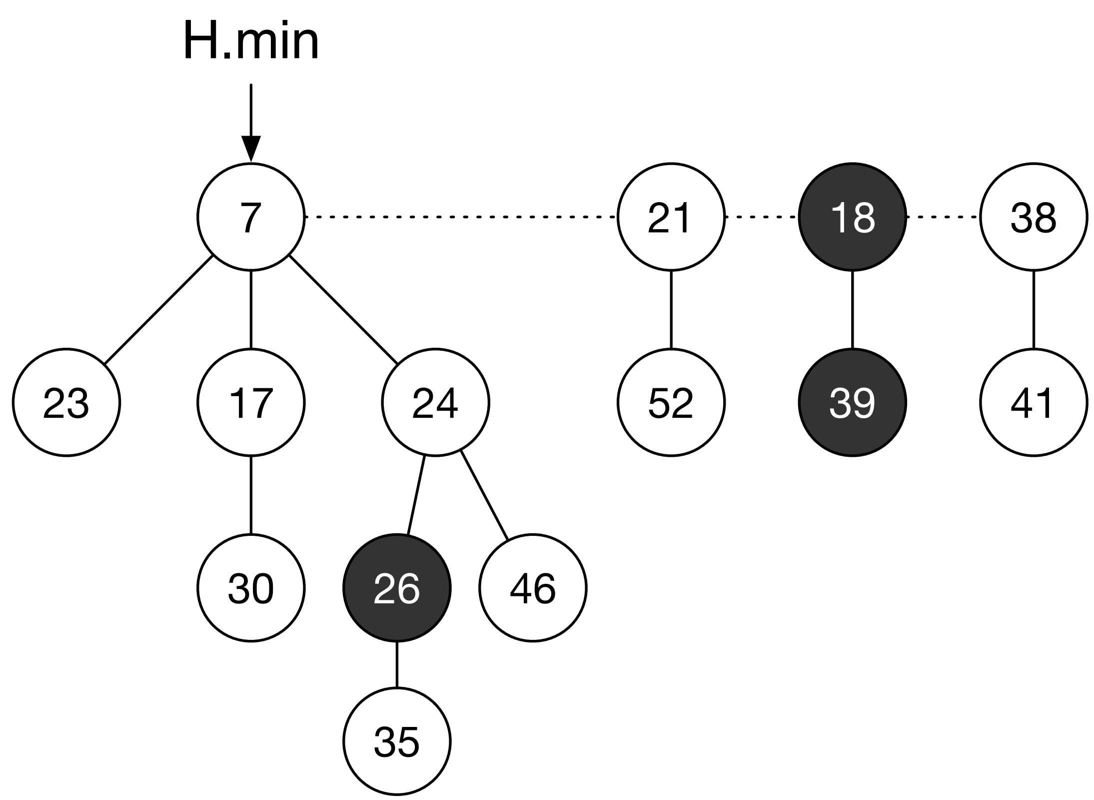
Fibonacci Heaps



A[0]: 21, 52
A[1]: 18
A[2]: NULL
A[3]: 7

Inserting 52 gives us another merge candidate

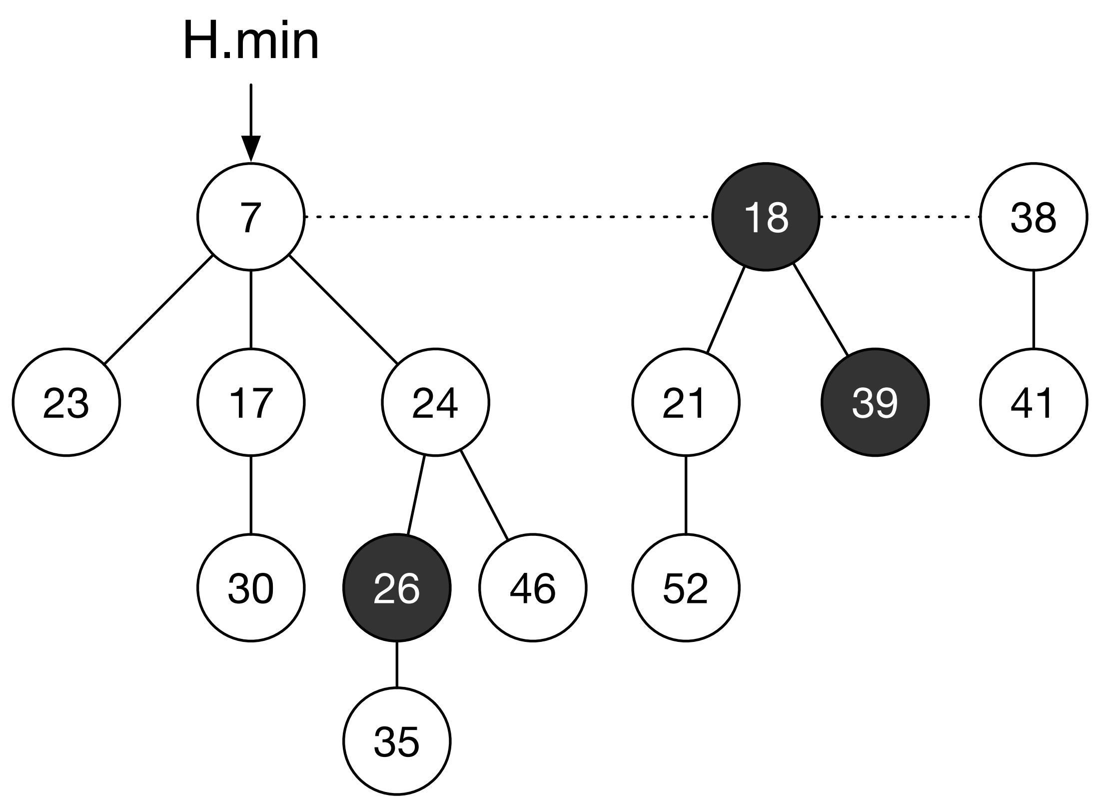
Fibonacci Heaps



A[0]: NULL
A[1]: 18, 21
A[2]: NULL
A[3]: 7

Which leads to another merger

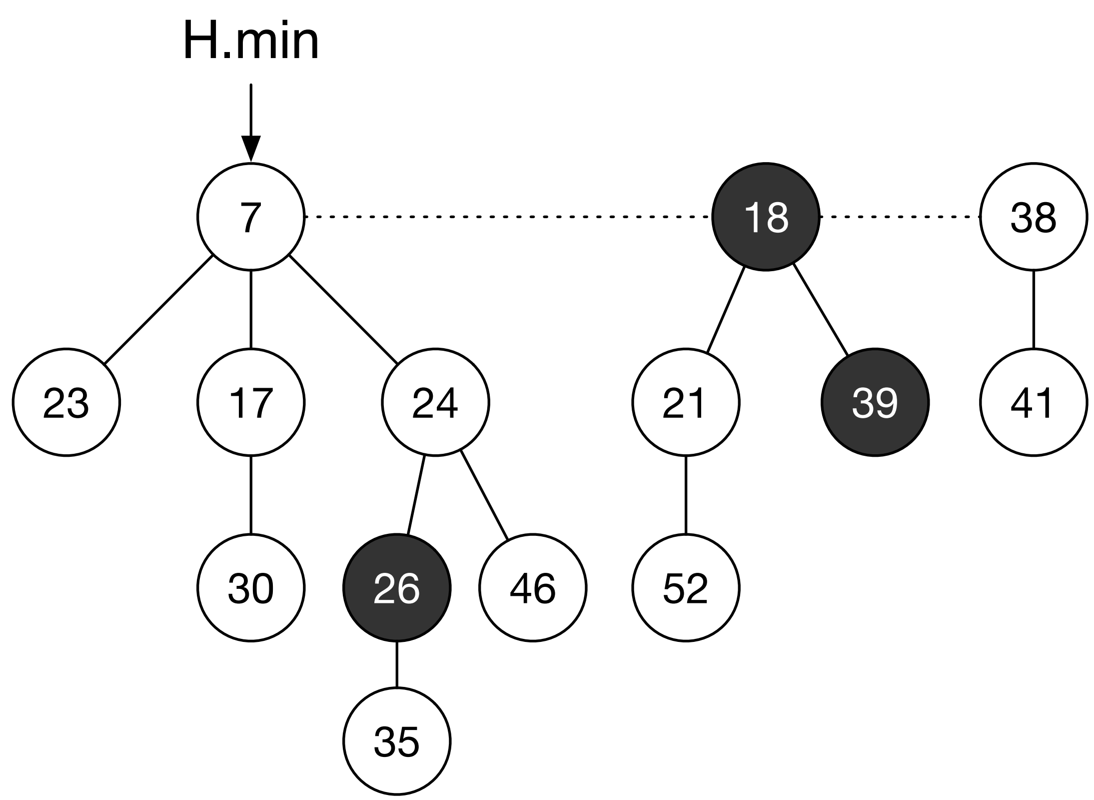
Fibonacci Heaps



A[0]: NULL
A[1]: NULL
A[2]: 18
A[3]: 7

We move on to the next node

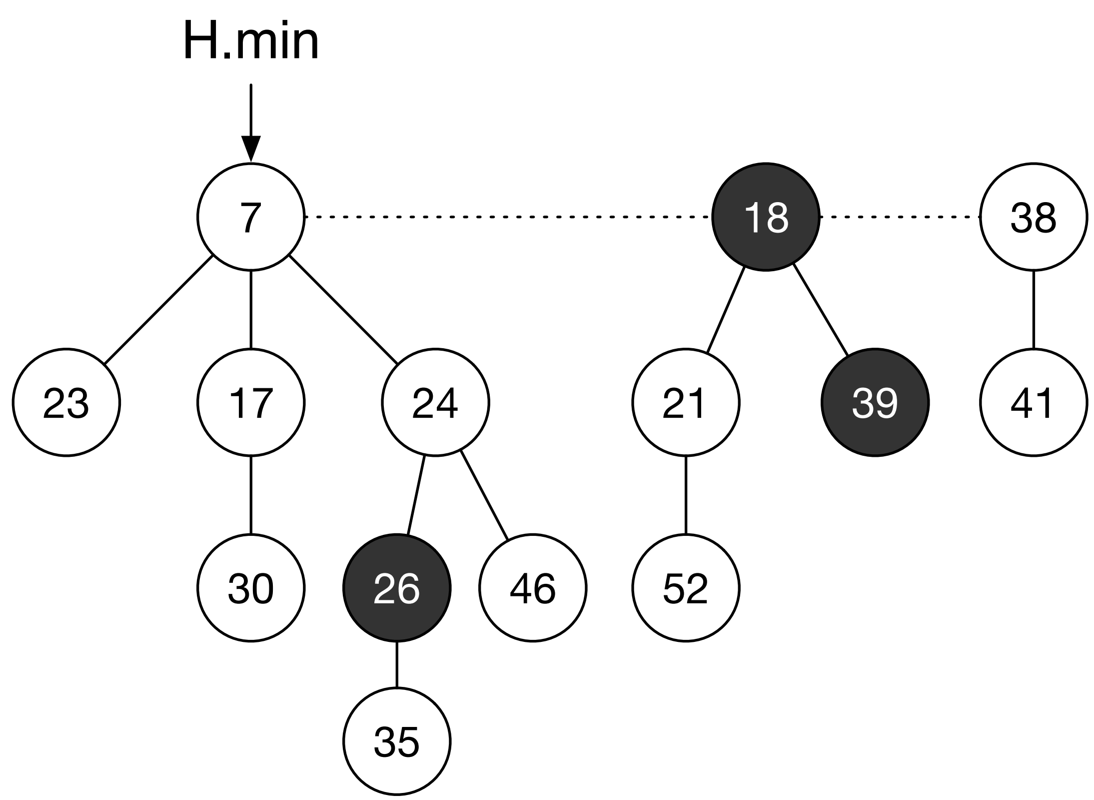
Fibonacci Heaps



A[0]: NULL
A[1]: 38
A[2]: 18
A[3]: 7

We insert it into the array.

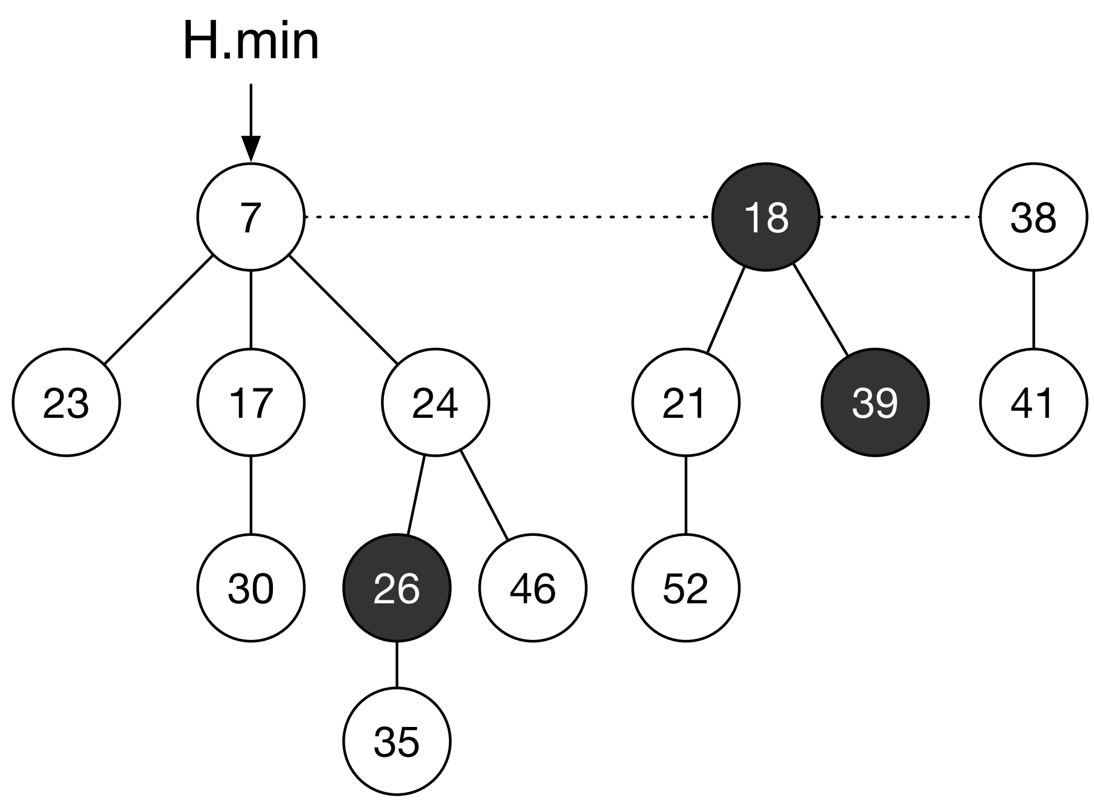
Fibonacci Heaps



A[0]: NULL
A[1]: 38
A[2]: 18
A[3]: 7,7

We then move on to the next node. When we insert, we see that this one is already in A and that we are done merging.

Fibonacci Heaps



A[0]: NULL
A[1]: 38
A[2]: 18
A[3]: 7

We now find the new minimum.

Fibonacci Heaps

- Combining two nodes
  - x.key < y.key

```
def fib_heap_link(H, y, x):
   Assert x.key < y.key
   remove y from root list of H
   y.mark = False
```

Fibonacci Heaps

- Consolidate:

```
def consolidate(H):
    A = [None for i in range(D(H.n))]
    for w in H.root_list:
        while(A[w.degree]):
            degree = A[w.degree]
            y = A[degree]  # candidate for merger
            if w.key > y.key:
                w, y = y, w
            fib_heap_link(H,y,w)
            A[degree] = Null
            degree += 1
        A[w.degree] = w
    reset H.min
```

Fibonacci Heaps

- Costs:
  - Potential:
    - Potential before is t(H)+2m(H)
    - Potential after is D(n)+1+2m(H)
      - because A has D(n) entries
      - because nobody gets marked
  - Work done:
    - t(H) for going through the root list
    - ≤ D(n) for inserting the children of minimum
  - Amortized costs:
  - $\leq O(D(n) + t(H)) + \big(D(n) + 1 + 2m(H) - t(H) - 2m(H)\big) = O(D(n))$

Fibonacci Heaps

- Decrease a key
    - Find parent p
        - if p and x.key < p.key:
            - CUT(H,x,y)
            - CASCADING-CUT(H,y)
    - If necessary, adjust H.min

Fibonacci Heaps

- Cut(H, x, y)

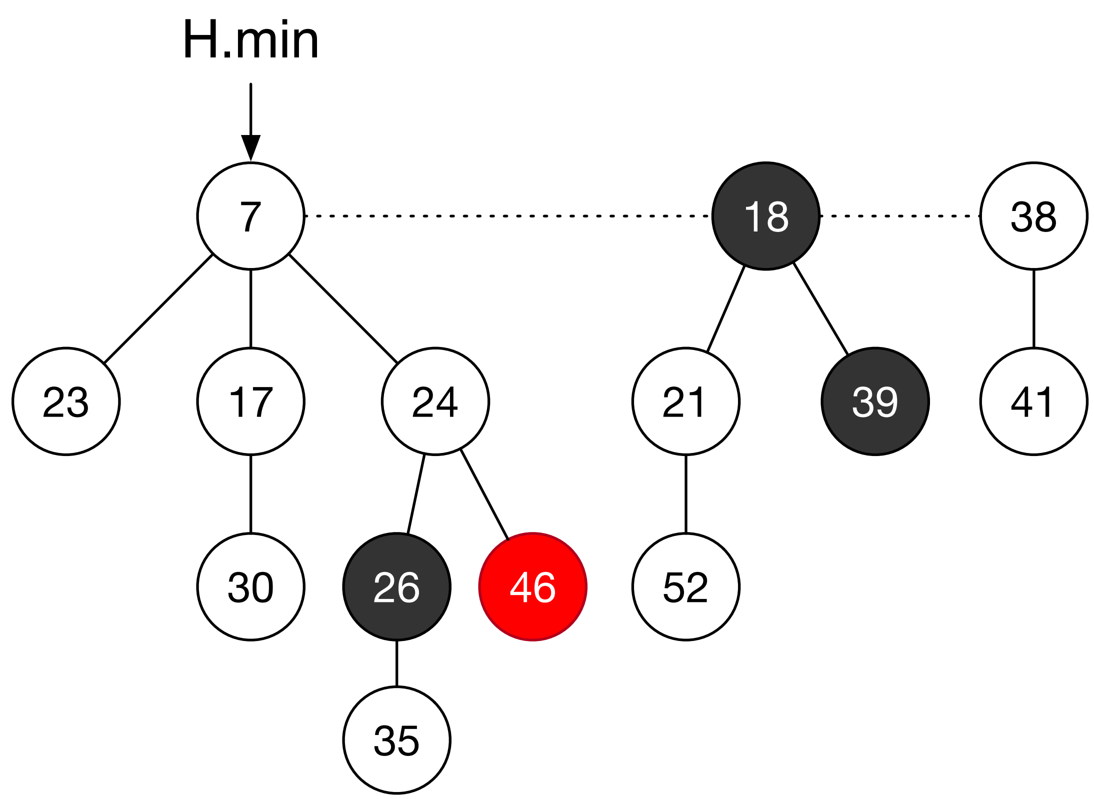
```
def cut(H, x, y):
   remove x from child list of y
   y.degree -= 1
   add x to root list of H
   x.p = Null
   x.mark = False
```

Fibonacci Heaps

- Cascading Cut

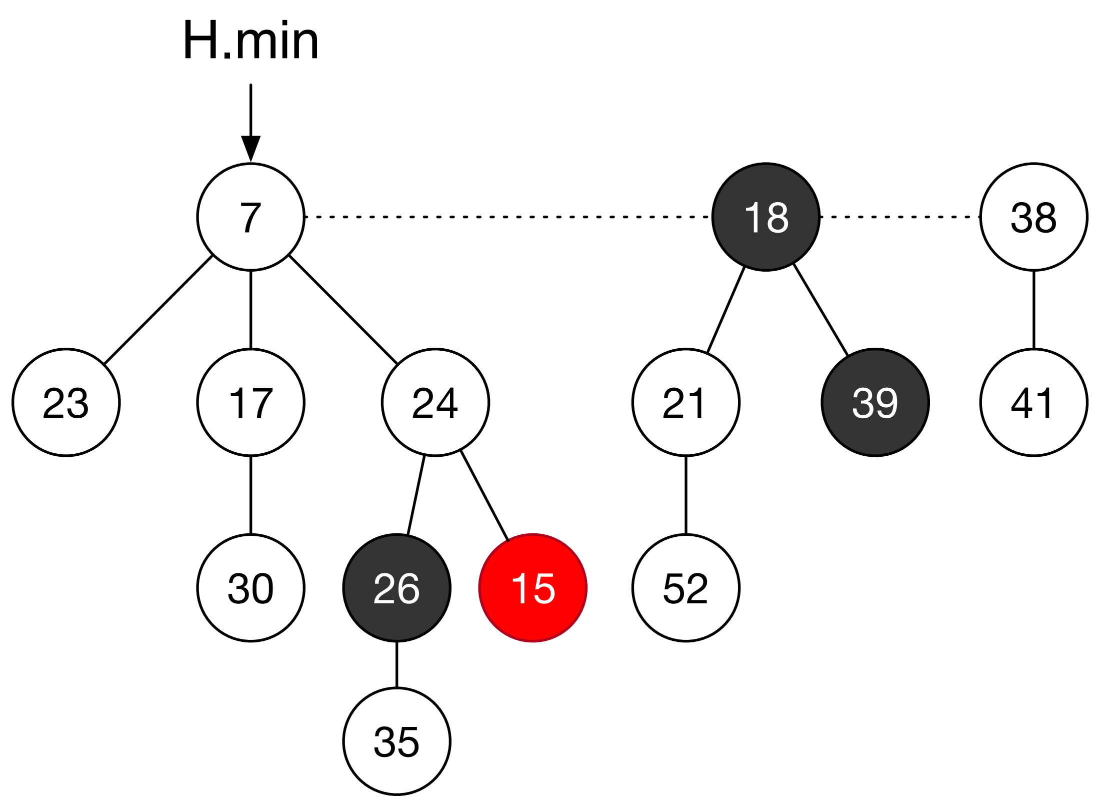
```
def cascading_cut(H,y):
    z = y.p # parent of y
    if z:
        if y.mark == False:
            y.mark = True
        else:
            cut(H, y, z)
            cascading_cut(H,z)
```

Fibonacci Heaps

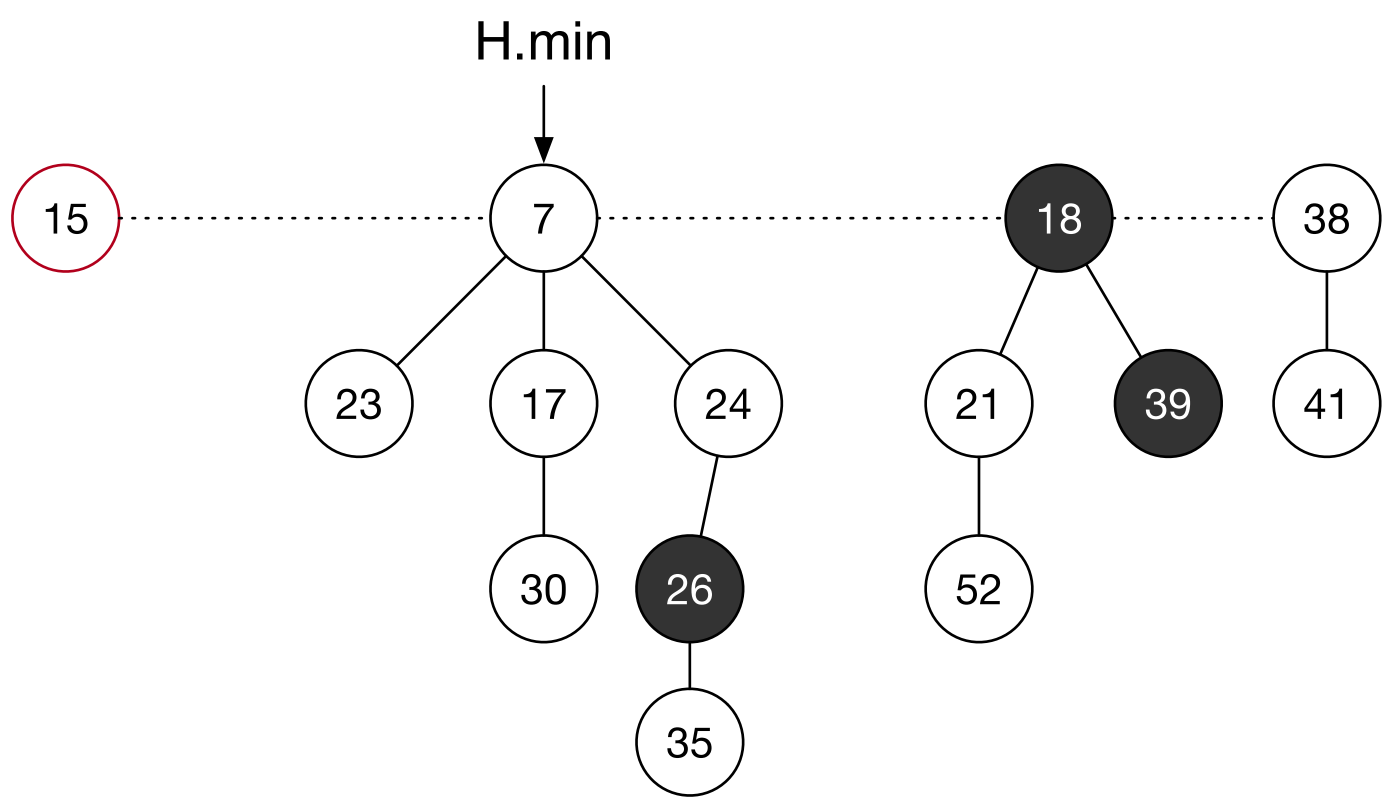


Decrease key to 15

Fibonacci Heaps

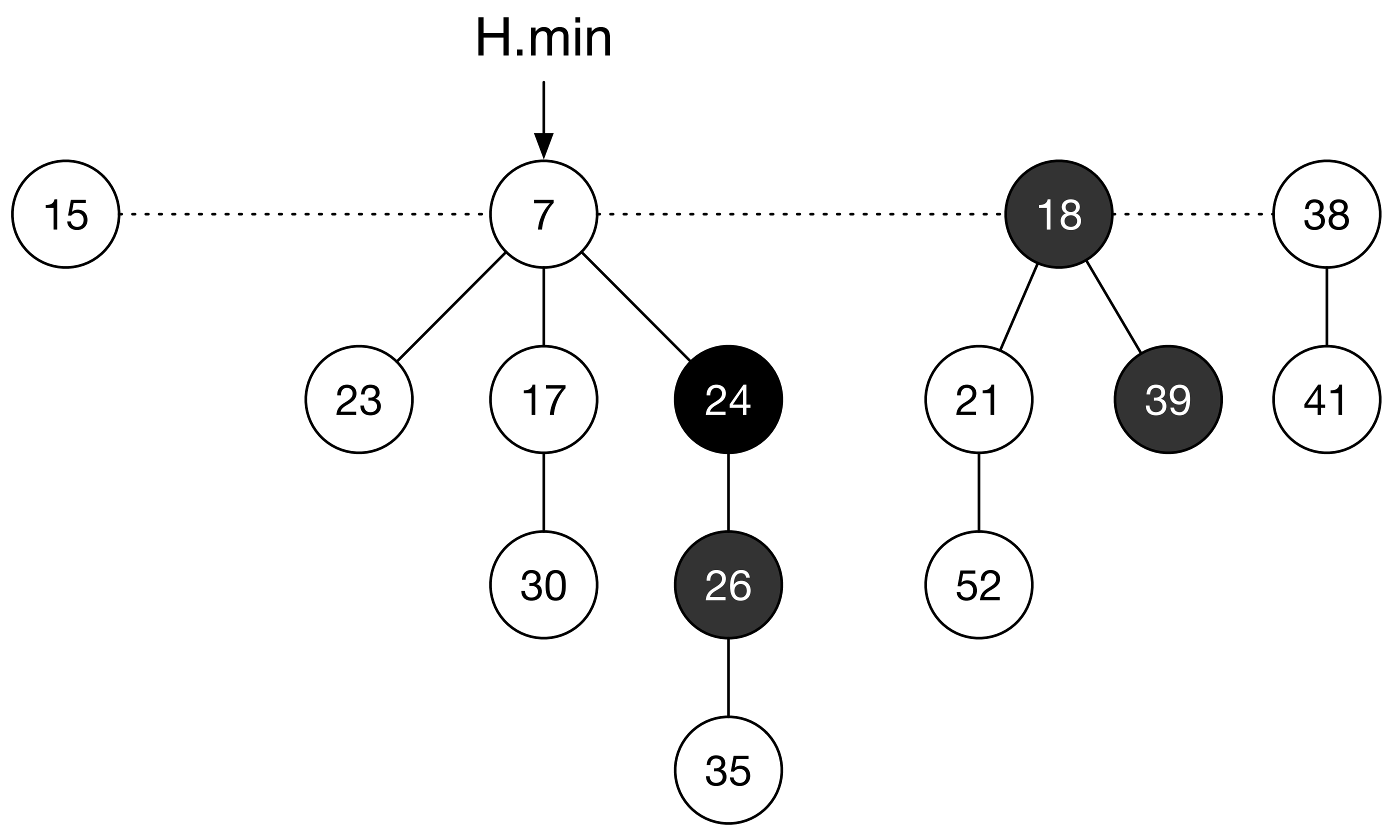


Fibonacci Heaps



Remove node (and any descendants)

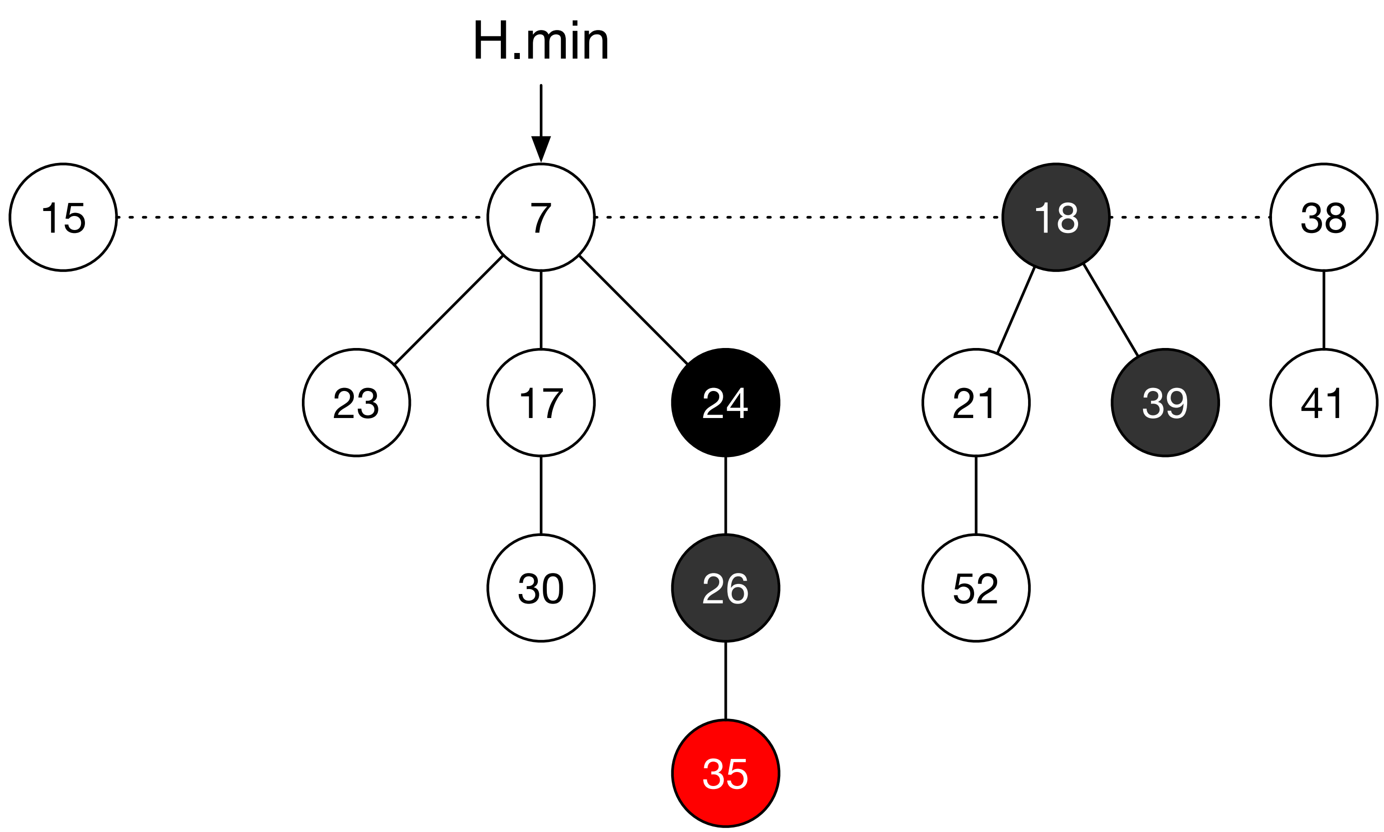
Fibonacci Heaps



Mark parent!
Done!

Fibonacci Heaps

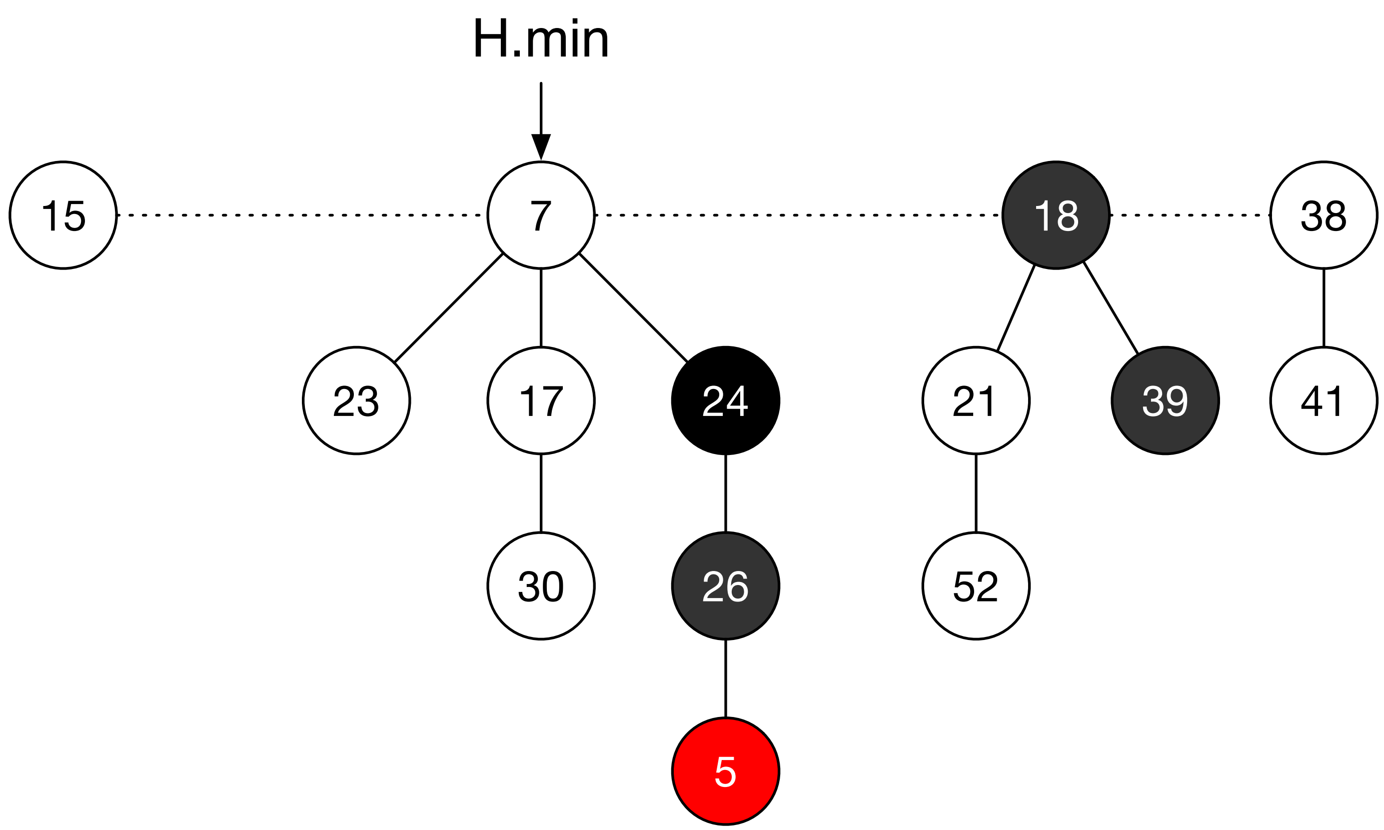
Second operation



Change key from 35 to 5

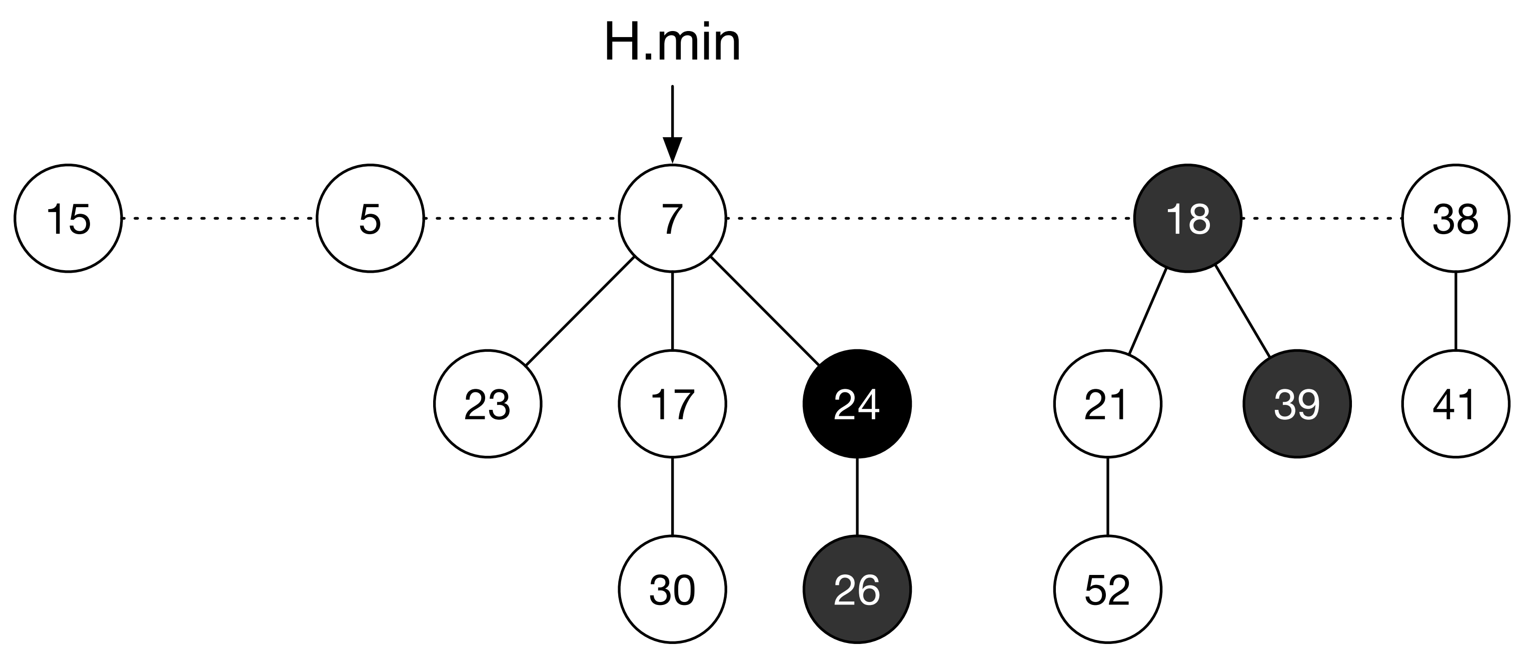
Fibonacci Heaps

Second operation



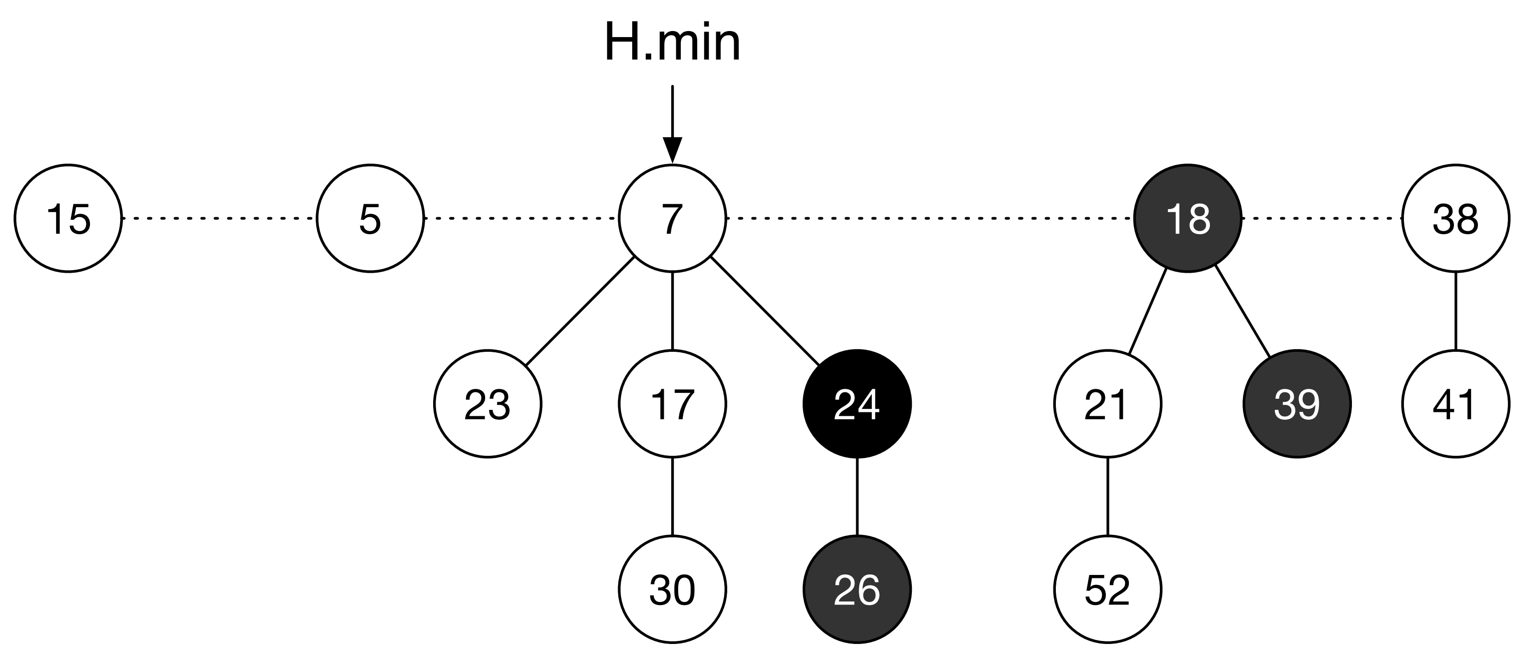
Change key from 35 to 5

Fibonacci Heaps



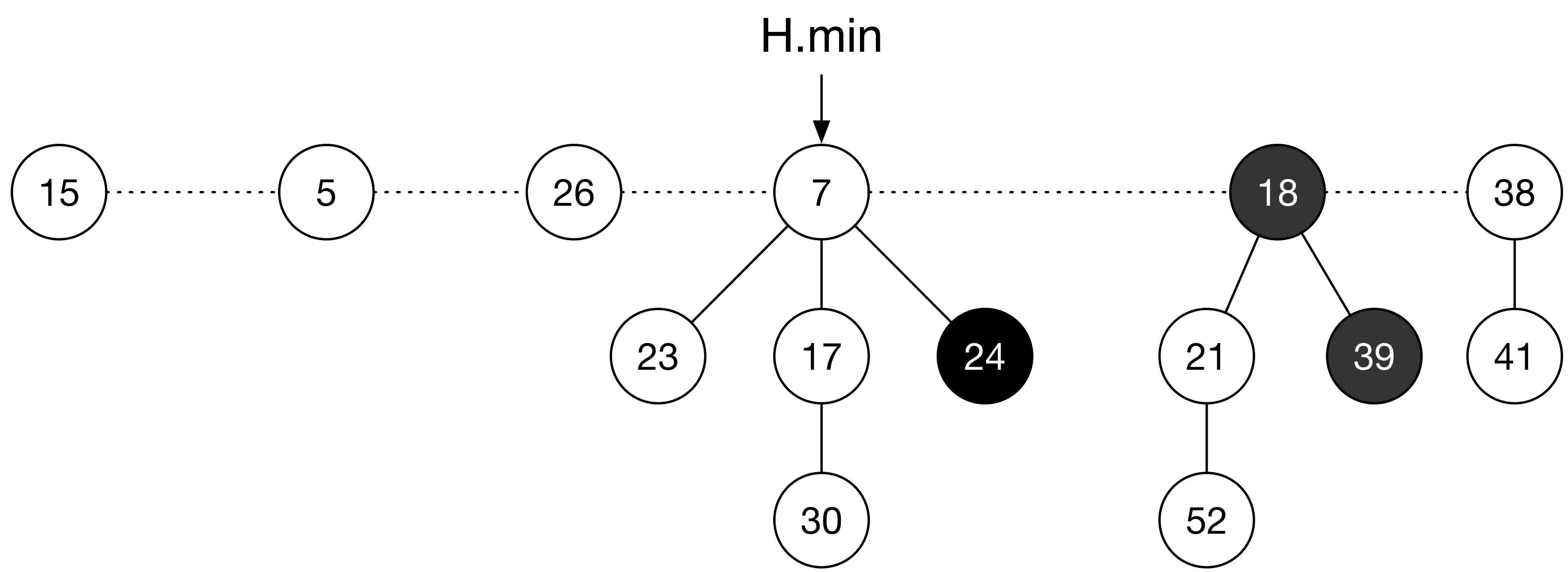
Cut node.

Fibonacci Heaps



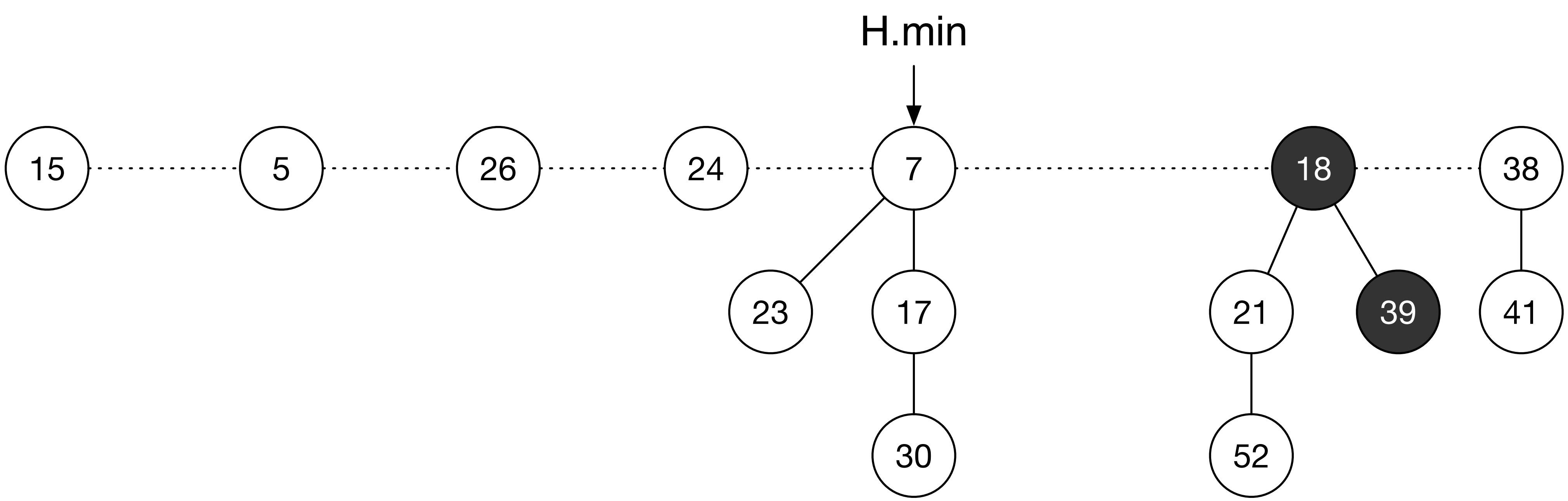
Mark parent.
But parent is already marked, so:
Cascading Cut!

Fibonacci Heaps



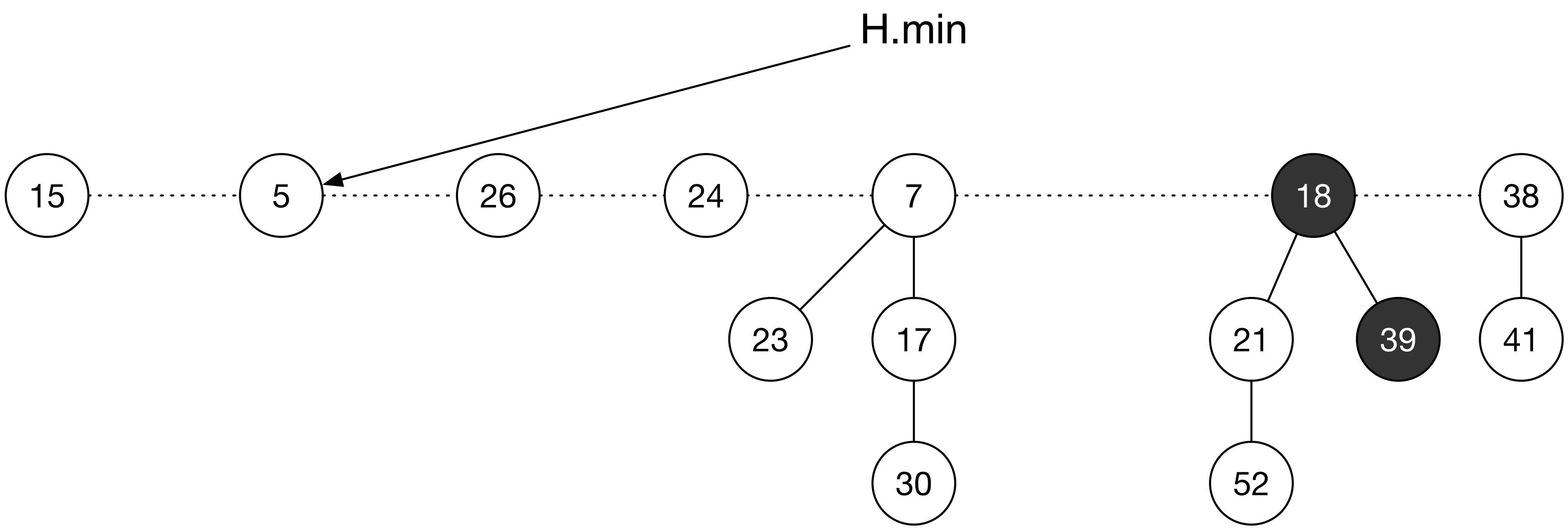
26 is removed.
Look at 24: It is also marked

Fibonacci Heaps



Cut 24, unmarking it.
Cut stops here, parent is unmarked.

Fibonacci Heaps



Reset minimum

Fibonacci Heaps

- Change in potential:
  - Cut creates a new tree and potentially clears a mark
  - Each Cascading-Cut cuts a marked node and clears the mark bit (with exception of the last one)
- H now has c-1 trees produced by cascading cuts and the tree at x: t(H)+c trees
- At most m(H)-c+2 marked nodes
- $\leq t(H) + c + 2(m(H) - c + 2) - t(H) - 2m(H) = 4 - c$

Fibonacci Heaps

- Amortized cost of decrease key:
  - $O(c) + 4 - c = O(1)$

Fibonacci Heaps

- Deleting a node
  -
    ```
    def delete(H,x):
        decrease_key(H,x,-infty)
        extract_min(H)
    ```

Fibonacci Heaps

- Left to investigate:
  - Upper bound *D*(*n*) on the degrees is $O(\log(n))$
- Lemma: $F_{k+2} = 1 + \sum_{i=0}^{k} F_i$

Fibonacci Heaps

- Lemma: $F_{k+2} \geq \phi^k$ with $\phi = \dfrac{1+\sqrt{5}}{2}$

Fibonacci Heaps

- Lemma: Let *x* be any node in a Fibonacci heap with x.degree = *k*. Let $y_1, y_2, \ldots, y_k$ be the children of *x* in the order in which they were linked to *x* (by consolidate). Then:

  y_1 . degree $\geq 0, \quad y_i$. degree $\geq i - 2$ for all $i = 2,3,\ldots,k$

Fibonacci Heaps

- Proof:
  - Clearly, y_1 . degree ≥ 0
  - Assume a general $i \geq 2$.
    - When CONSOLIDATE links y_i to x, then all of $y_1, y_2, \ldots$ y_{i-1} was linked to x
    - This means x . degree $\geq i$
    - Because y_i is linked to x only if x . degree $= y_i$. degree
      - y_i . degree $\geq i - 1$

Fibonacci Heaps

- But in the mean-time, the degree of y_i might have decreased
- But Cascading-Cut would cut y_i from x if y_i has lost more than two children
- Therefore y_i . degree $\geq i - 2$

Fibonacci Heaps

- Lemma: Let x be any node in a Fibonacci heap and let $k = x$. degree. Then size$(x) \geq F_{k+2} \geq \Phi^k$.
    - Let s_k denote the minimum possible size of any node of degree k in a Fibonacci heap.
    - $s_0 = 1, \qquad s_1 = 2$
    - Adding children does not decrease the node's size, the value of s_k increases monotonically with *k.*

Fibonacci Heaps

- Notice $s_k \leq \text{size}(x)$, so giving a lower bound on s_k is sufficient
- Take a node z with
  - $z\,.\,\text{degree} = k \qquad \text{size}(z) = s_k$
- Let $y_1, y_2, \ldots, y_k$ denote the children of z in the order in which they were linked to z
- To bound s_k, we have one for z and one for y_1 and the rest

Fibonacci Heaps

- This gives
  - size$(x) \geq s_k$
  - $\geq 2 + \sum_{i=2}^{k} s_{y_i.\text{degree}}$
  - $\geq 2 + \sum_{i=2}^{k} s_{i-2}$ because y_i . degree $\geq i - 2$ by Lemma

    and monotonicity of s_k

Fibonacci Heaps

- We prove by induction that $s_i \geq F_{i+2}$
- Induction step:
- $s_k \geq 2 + \sum_{i=2}^{k} s_{i-2}$
- $\geq 2 + \sum_{i=2}^{k} F_i$
- $= 1 + \sum_{i=0}^{k} F_i$
- $=F_{k+2} \geq \Phi^k$

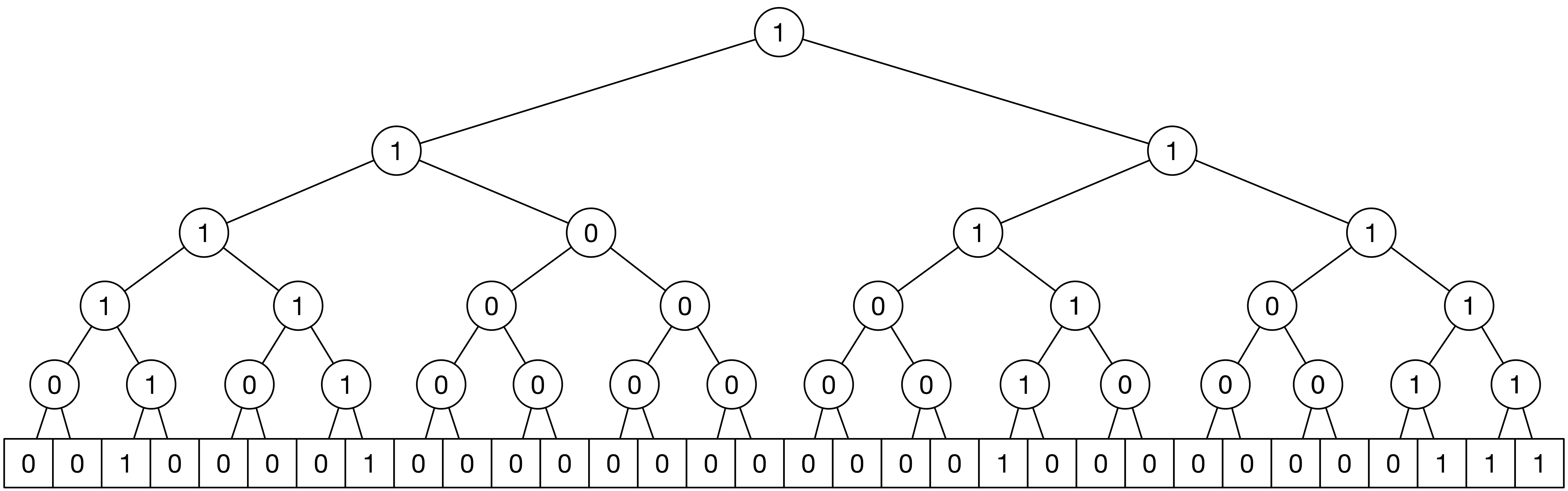
Fibonacci Heaps

- Let x be the node with maximum degree into an n-node Fibonacci Heap
- Degree of x is k
- By lemma, $n \geq \text{size}(x) \geq \Phi^k$
- Thus: $\log_\Phi(n) \geq k$
- Thus: maximum degree is $O(\log n)$

van Emde Boas Trees

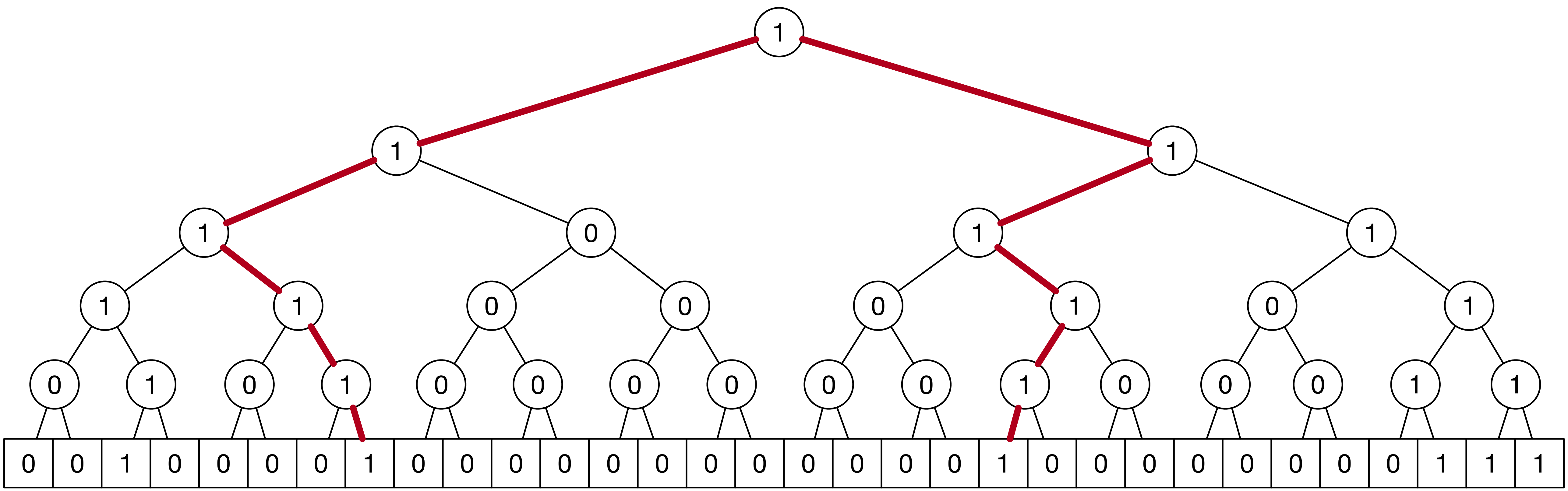
- Operations of priority queues have at least one logarithmic operation
- If everything would be less, then we could sort in time $n \cdot \log n$
- But we can sort an array of integers in range 1 … *n* in time O(*n*)

van Emde Boas Trees



Binary tree on top of a bitvector

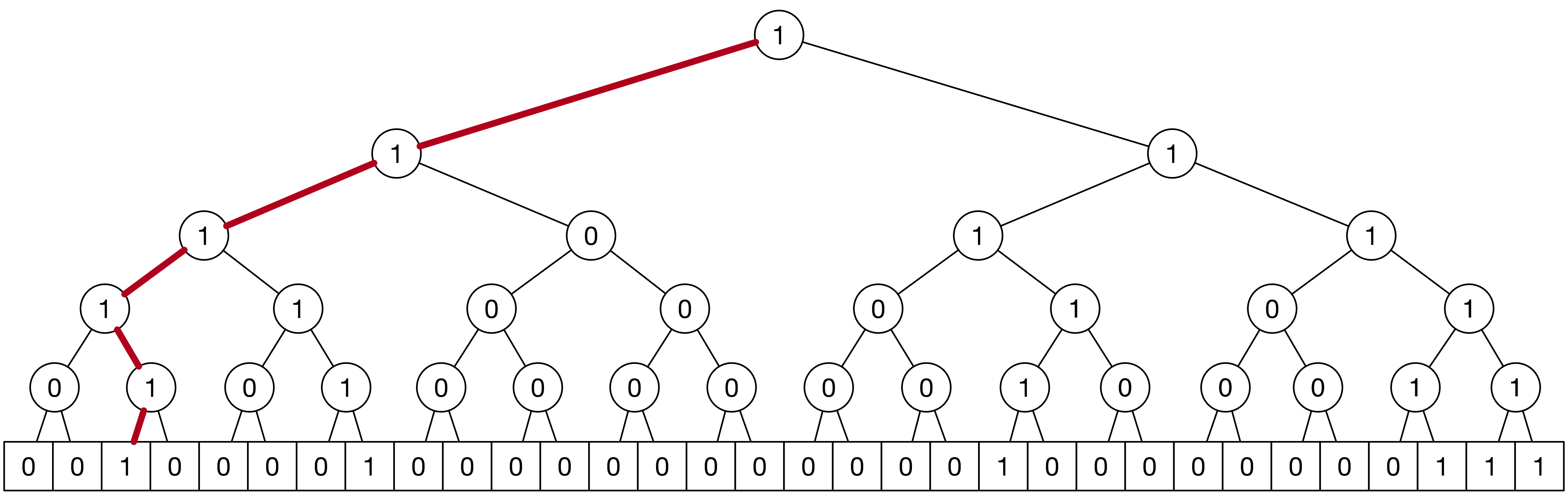
van Emde Boas Trees



Finding the NextIndex / PreviousIndex
is logarithmic

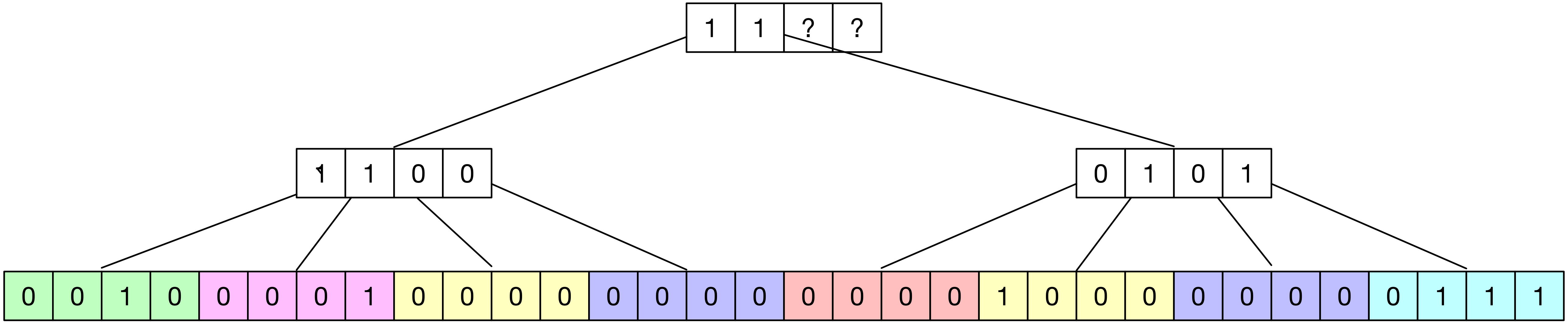
van Emde Boas Trees

- Finding the minimum value in logarithmic time:
  - Start at root
  - Head down taking the leftmost one



van Emde Boas Trees

- We can combine nodes into a tree with higher fan-out

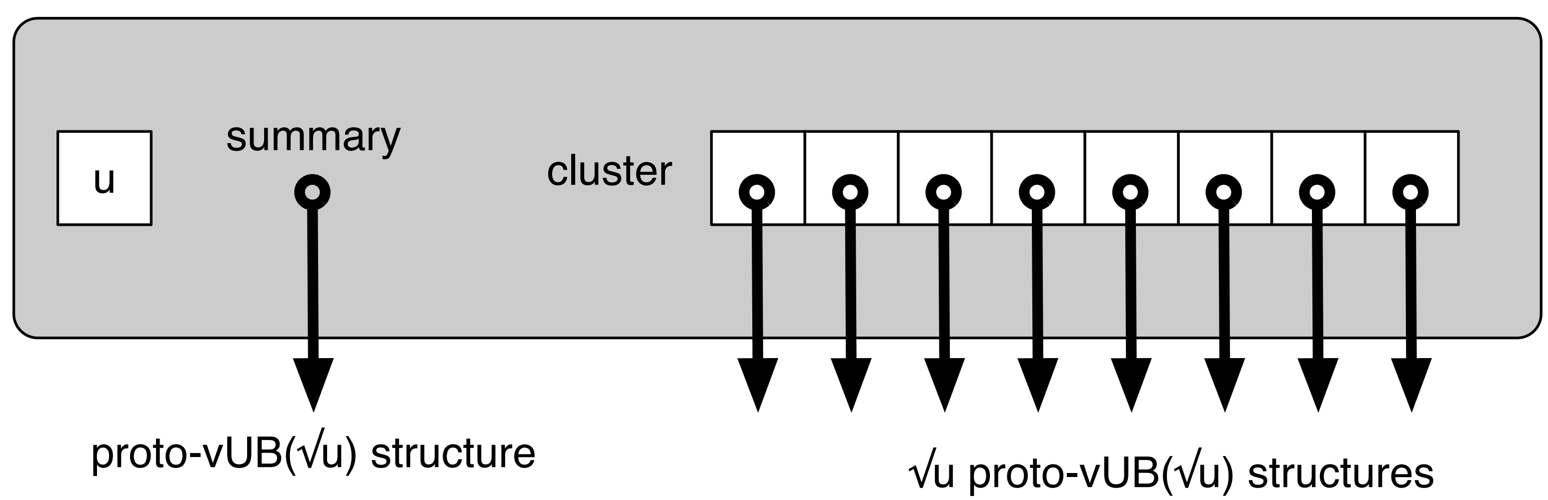


van Emde Boas Trees

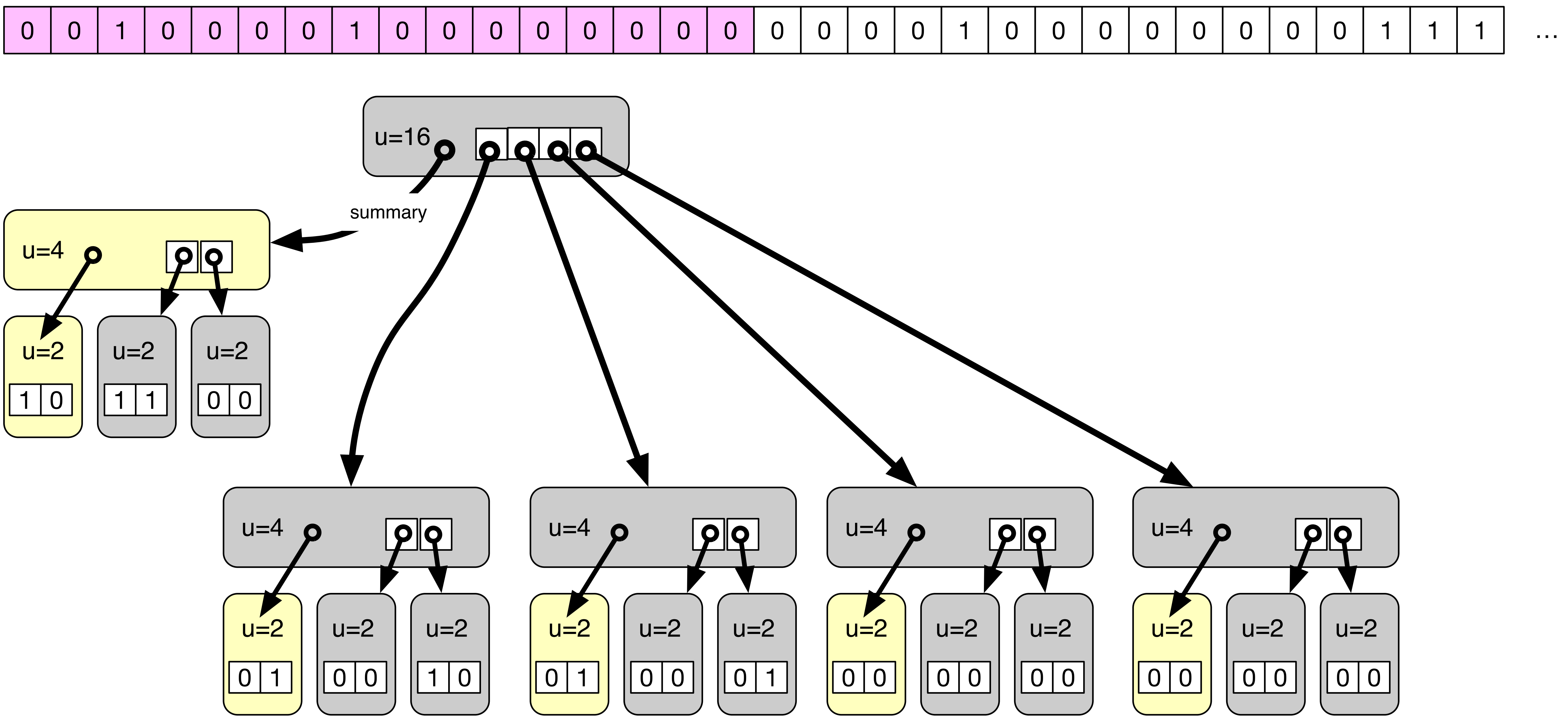
- Assume that the number u of elements in the universe is $u = 2^{2^k}$
- To build a recursive structure:
  - Recurrence $T(u) = T(\sqrt{u}) + O(1)$
    - Setting $m = \log_2(u)$, $S(m) = T(2^m)$
    - Recurrence is $S(m) = S(m/2) + O(1)$
    - MT $\implies S(m) = O(\log_2(m))$
    - $T(u) = T(2^m) = S(m) = O(\log_2(m)) = O(\log_2(\log_2(u)))$

van Emde Boas Trees

- Proto-vEM-structures for u
  - $u = 2$: array of two bits
  - $u = 2^{2^k}$ with $k \geq 1$:
    - summary pointer towards parent
    - cluster towards children



van Emde Boas Trees



van Emde Boas Trees

- Operations on proto-vEB-Trees
  - For a given $x \in U$:
    - $\text{high}(x) = \lfloor \frac{x}{\sqrt{u}} \rfloor$ number of cluster
    - $\text{low}(x) = x \pmod{\sqrt{u}}$ position of x in cluster
    - $\text{index}(x, y) = x\sqrt{u} + y$ rebuilds index from number of cluster and position in cluster

van Emde Boas Trees

- Operations on proto-vEB-Trees
  - Membership

```
def pvEB_member(V,x):
    if V.u == 2:
        return V.A[x]
    else:
        return pvEB_member(V.cluster[high(x)].low(x)
```

  - Runtime has recurrence $T(u) = T(\sqrt{u}) + O(1)$

van Emde Boas Trees

- Operations on proto-vEB-Trees
    -

van Emde Boas Trees

- Operations on proto-vEB-Trees

van Emde Boas Trees

- Operations on proto-vEB-Trees

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- Operations on proto-vEB-Trees

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