Graph Algorithms

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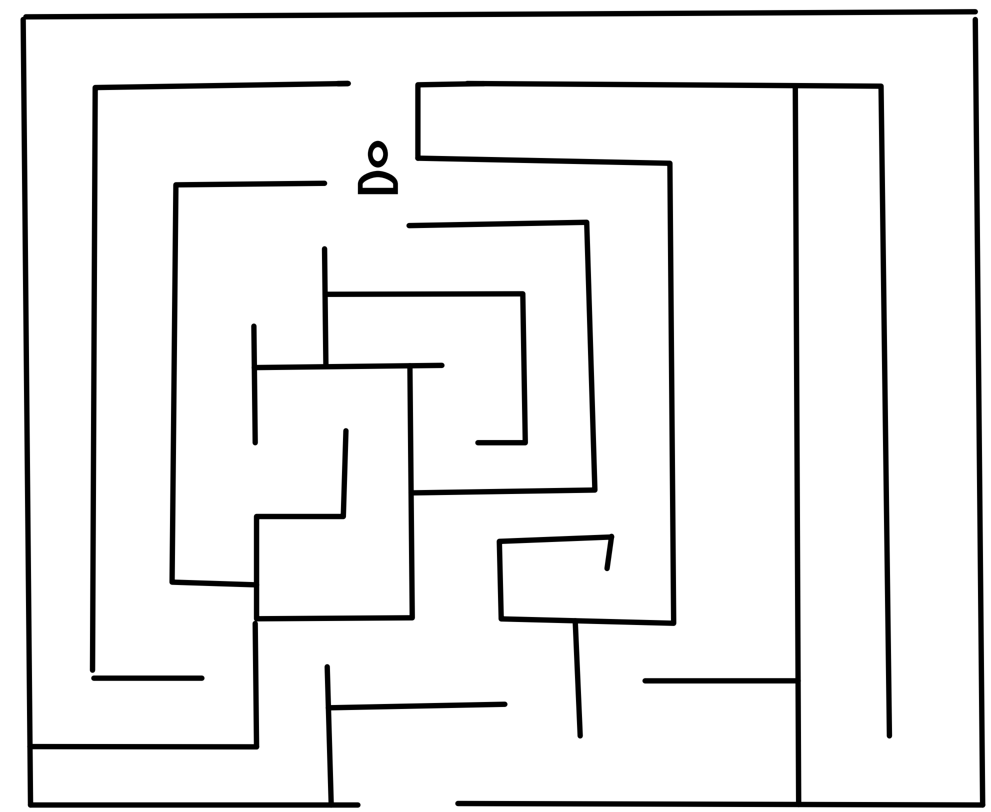
Searching in Graphs

- Exploring a maze
  - You are in the middle of a maze
    - How do you get out
  - Ariadne's solution:
    - Use a thread of glittering jewels in order to avoid using the same edges several times
  - Follow a wall
    - Works for simple mazes

Trémaux's Algorithm

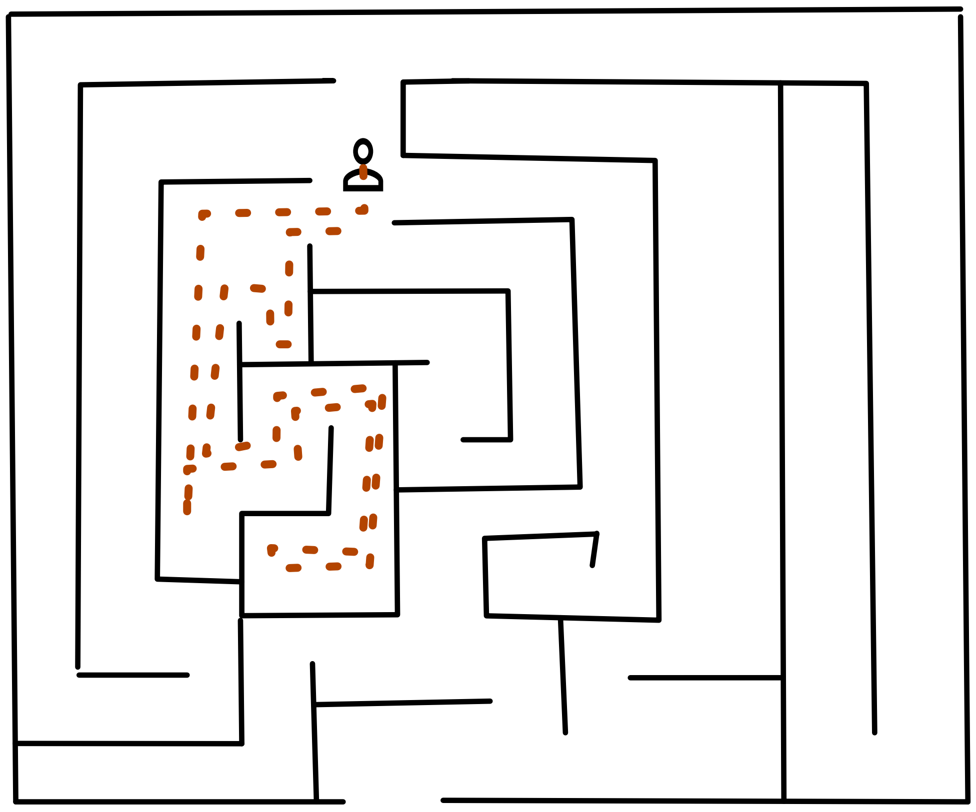
- Trémaux's Algorithm aka Hansel and Gretel's aka Ariadne's
  - Carry bread and leave bread crumbs on each path you follow
  - If you come to an intersection, follow one where there are no bread crumbs, if you can
  - If you come to an intersection and everything has already been marked or you are at a dead-end, turn around if you came at a path that has only one thread of crumbs



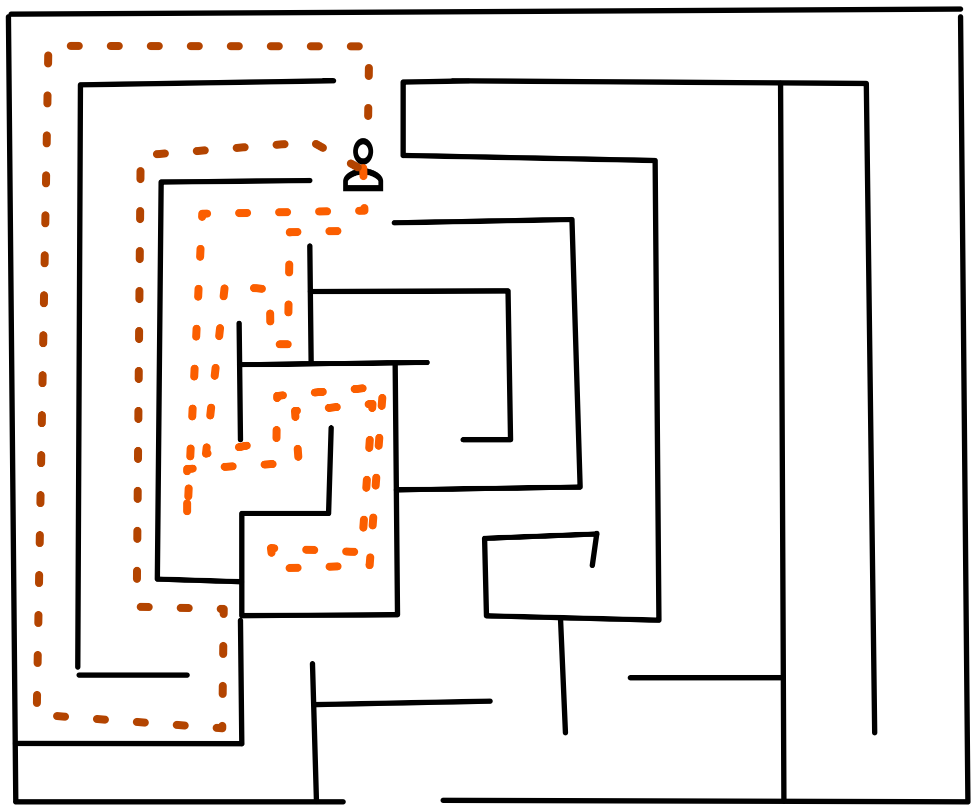
- If not, follow a path that has only one trail of crumbs.

Trémaux's Algorithm

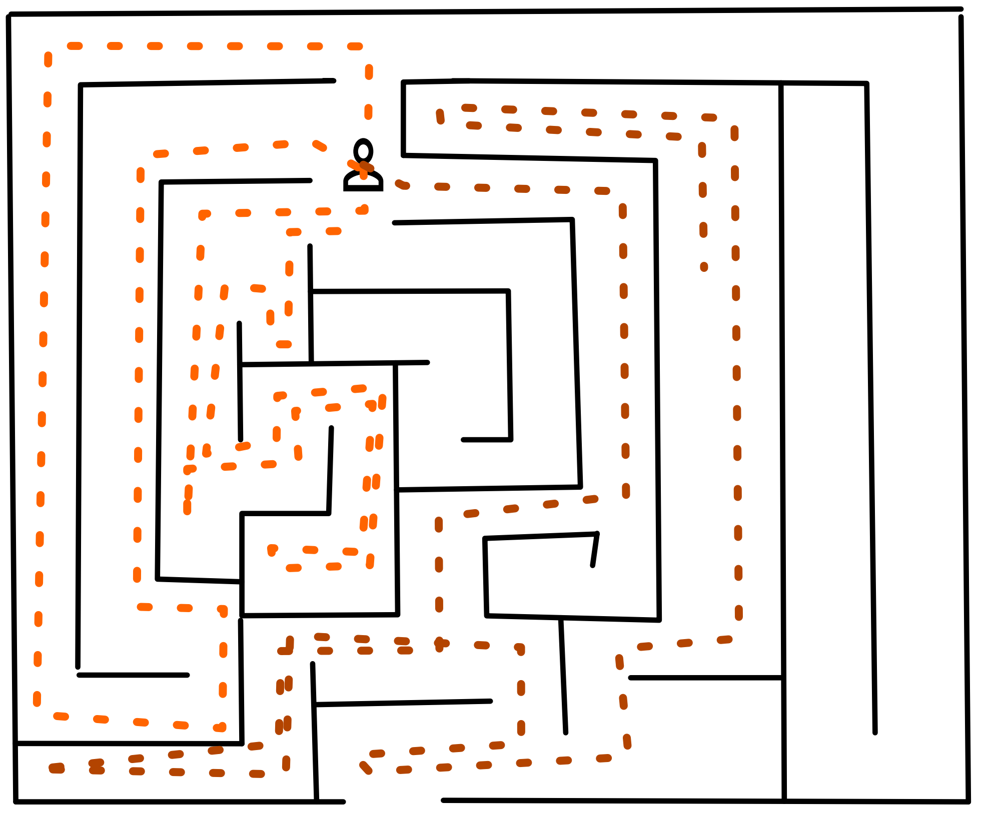
- Example



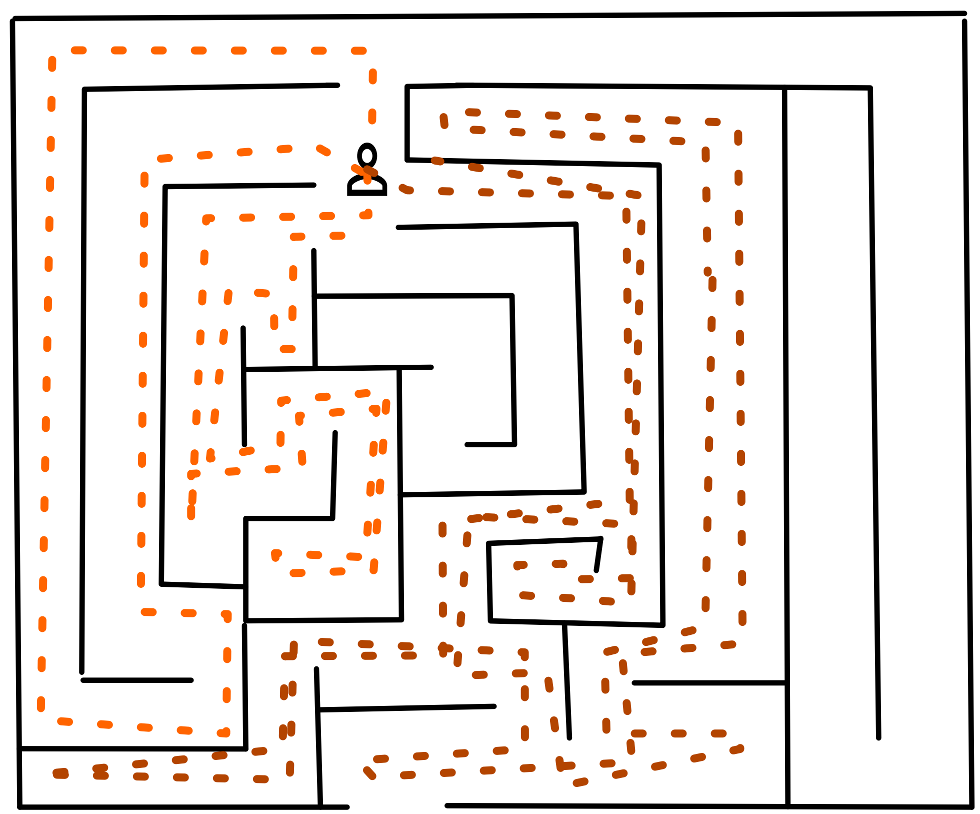
Trémaux's Algorithm



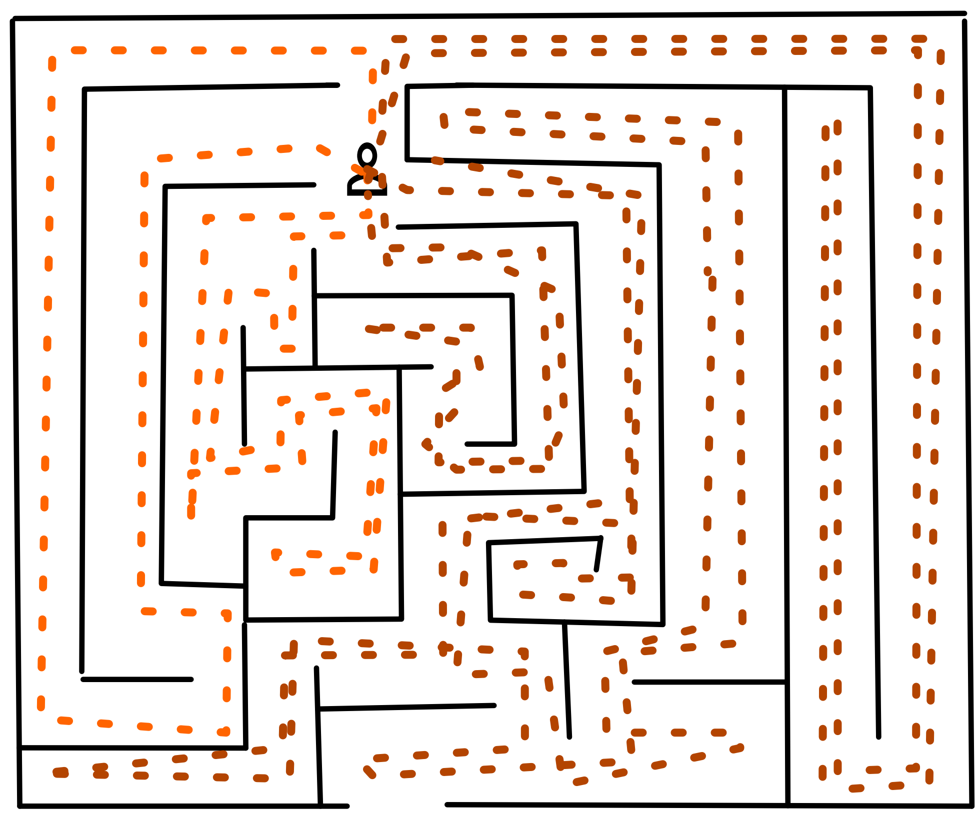
Trémaux's Algorithm



Trémaux's Algorithm



Trémaux's Algorithm



Trémaux's Algorithm

- https://en.wikipedia.org/wiki/File:Tremaux_Maze_Solving_Algorithm.gif

Trémaux's Algorithm

- At the end:
  - All paths will be double marked and you will end up at the starting point
  - This means that you walked by the entry

Searching in Graphs

- We can use this idea for defining the first graph exploration algorithm.
  - Goal is to visit all vertices
  - We use a timer:
    - Starts out at 0
    - Incremented every time we do something
  - All nodes get marked with a
    - Discovery time: First time that we see the node
    - Finishing time: When we are done with the node

Breadth First Search

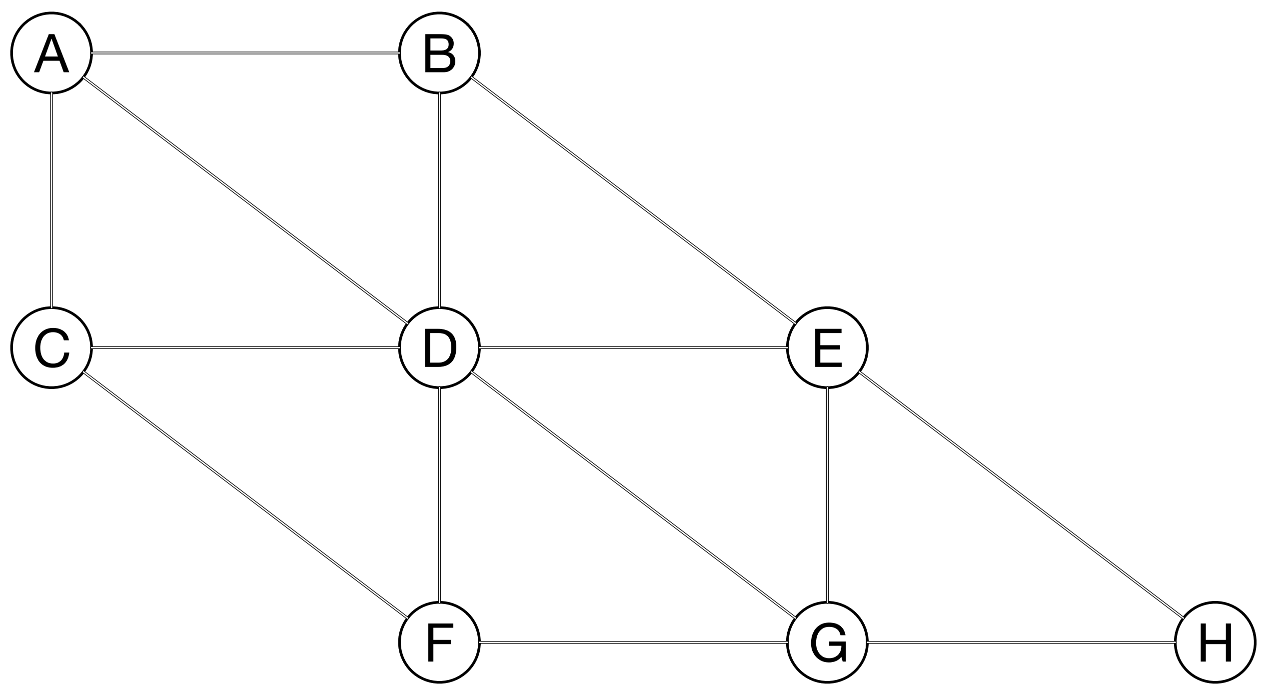
- Color a vertex
  - white: vertex has not yet been discovered
  - gray: vertex has been discovered, but still needs to be a base for exploration
  - black: vertex has been dealt with

Breadth First Search

```
bfs(G,s):
   for v in G.vertices:
      v.color = 'white'
      v.dist = inf
      v.pred = None
    s.color = 'gray'
    s.dist = 0
    s.pred = None
    queue = []
    queue.append(s)
    while queue:
       u = queue.pop(0)
       for v in u.adjacency:
          if v.color == 'white'
              v.color = 'gray'
              v.dist = u.dist+1
              v.pred = u
              queue.append(v)
       u.color = 'black'
```

Breadth First Search

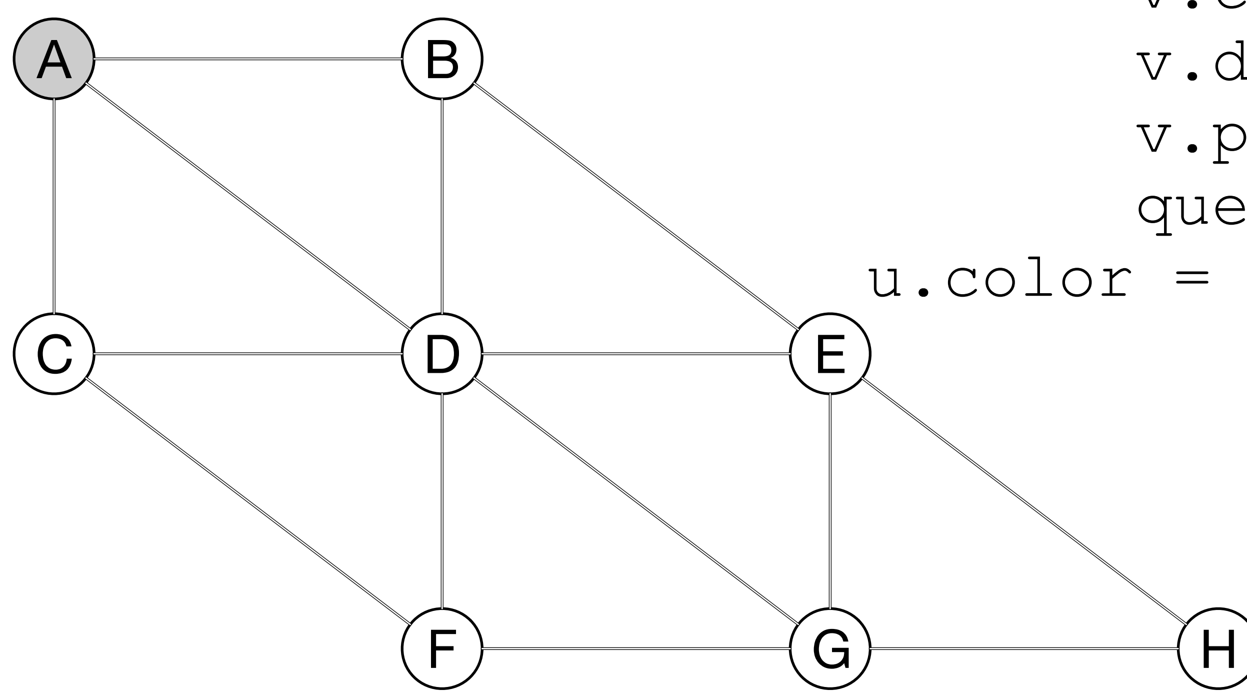
- Example: s=A



Breadth First Search

- queue = {A}

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
           v.dist = u.dist+1
           v.pred = u
           queue.append(v)
    u.color = 'black'
```



Breadth First Search

- queue = { }
- u = A

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
           v.dist = u.dist+1
           v.pred = u
           queue.append(v)
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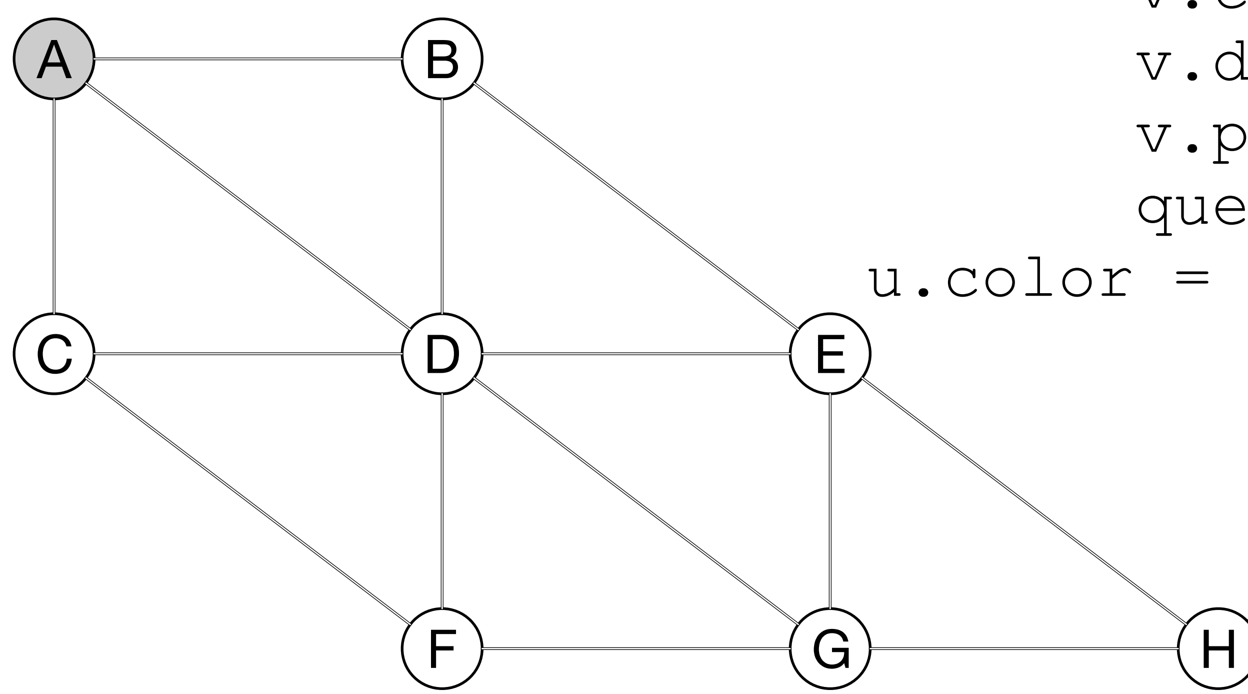

Breadth First Search

- queue = { }
- u = A

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
           v.dist = u.dist+1
           v.pred = u
           queue.append(v)
    u.color = 'black'
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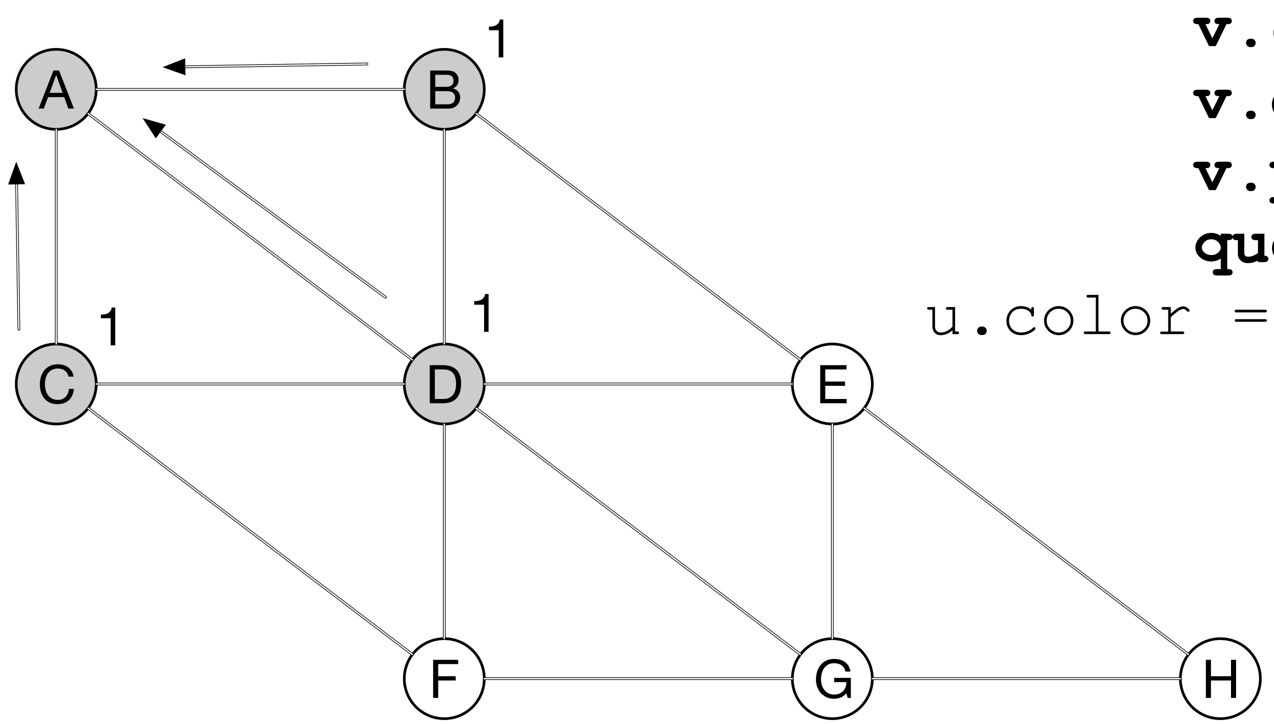
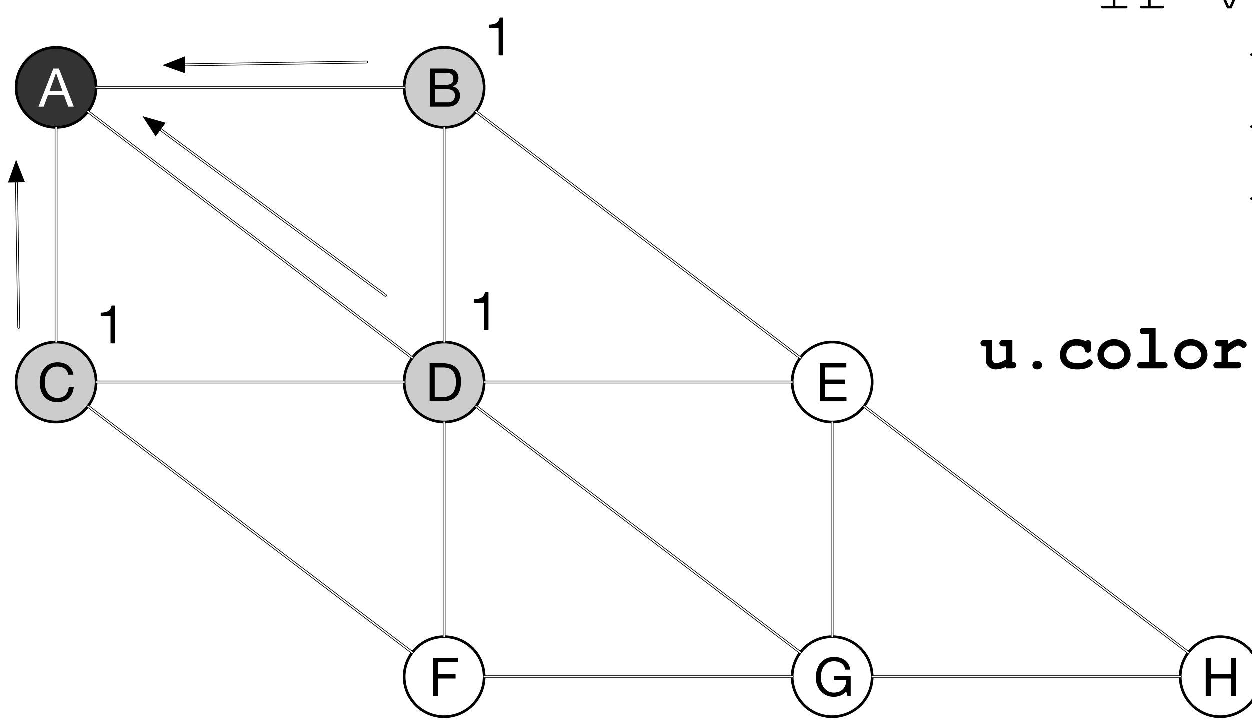


- queue = {B,C,D}

Breadth First Search

- queue = { }
- u = A

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
           v.dist = u.dist+1
           v.pred = u
           queue.append(v)
    u.color = 'black'
```

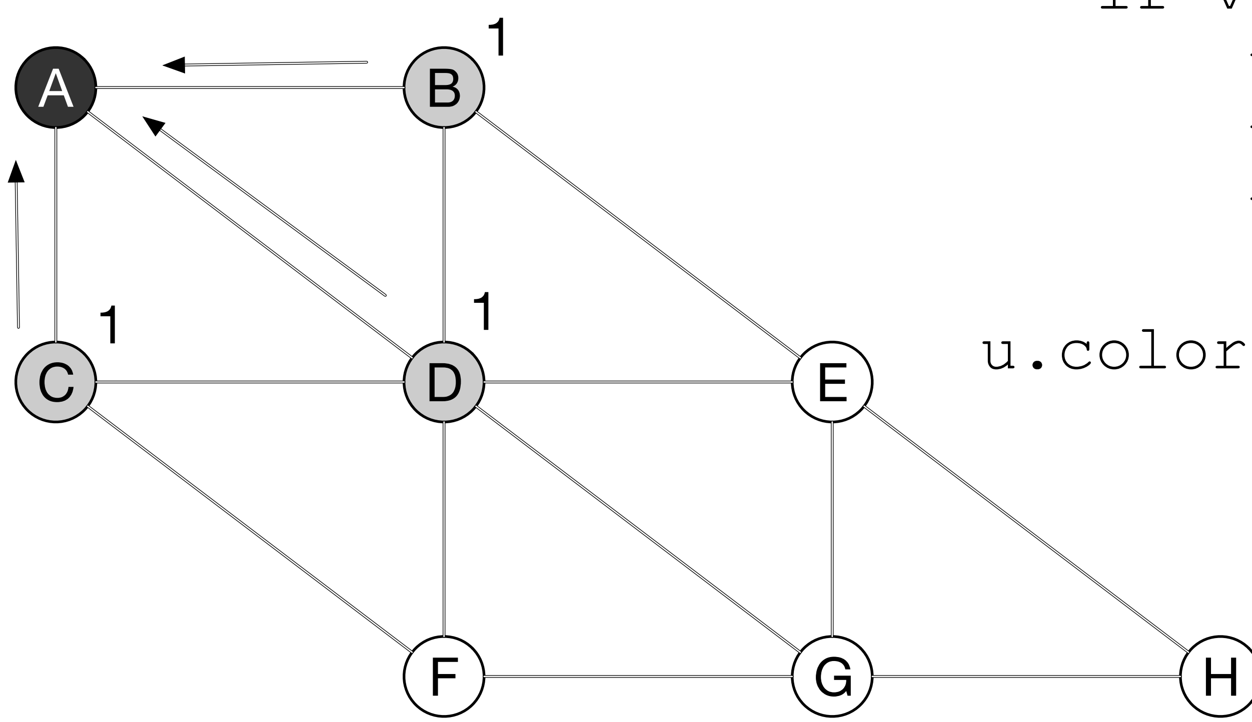


- queue = {B,C,D}

Breadth First Search

- queue = {C,D }
- u = B

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
        if v.color == 'white'
            v.color = 'gray'
            v.dist = u.dist+1
            v.pred = u
            queue.append(v)
    u.color = 'black'
```

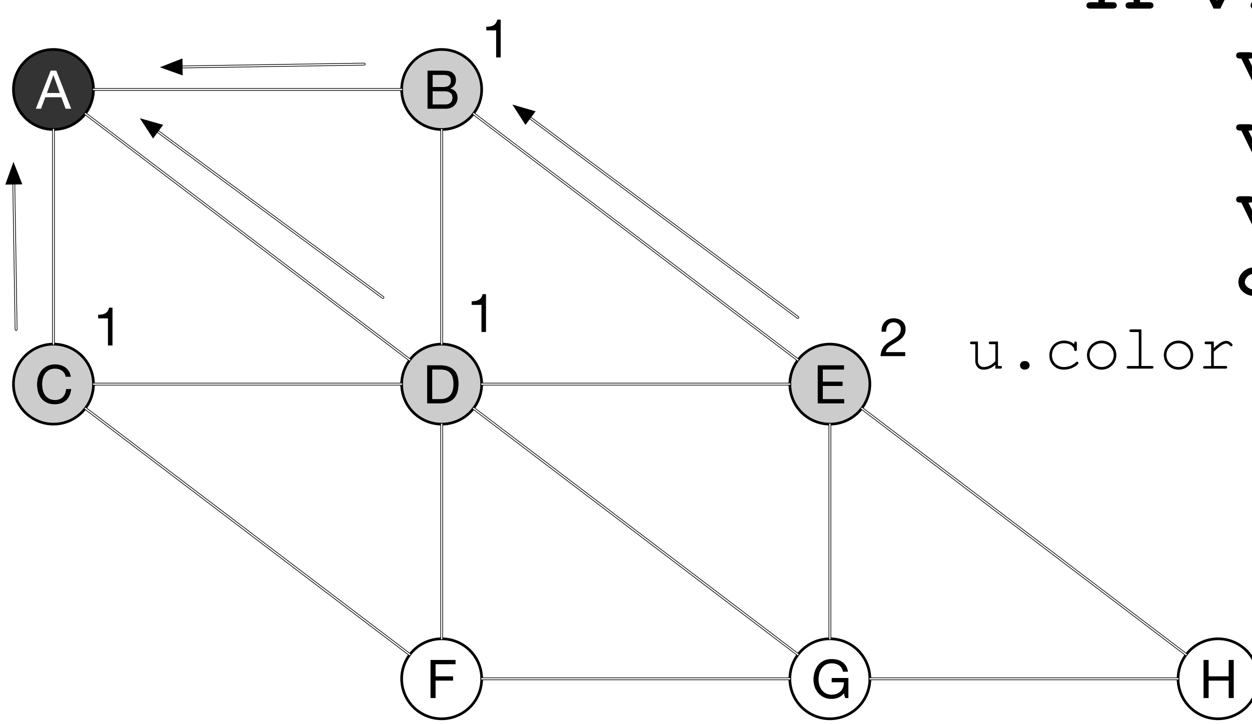


- queue = {C,D}

Breadth First Search

- queue = {C,D }
- u = B

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
           v.dist = u.dist+1
           v.pred = u
           queue.append(v)
    u.color = 'black'
```

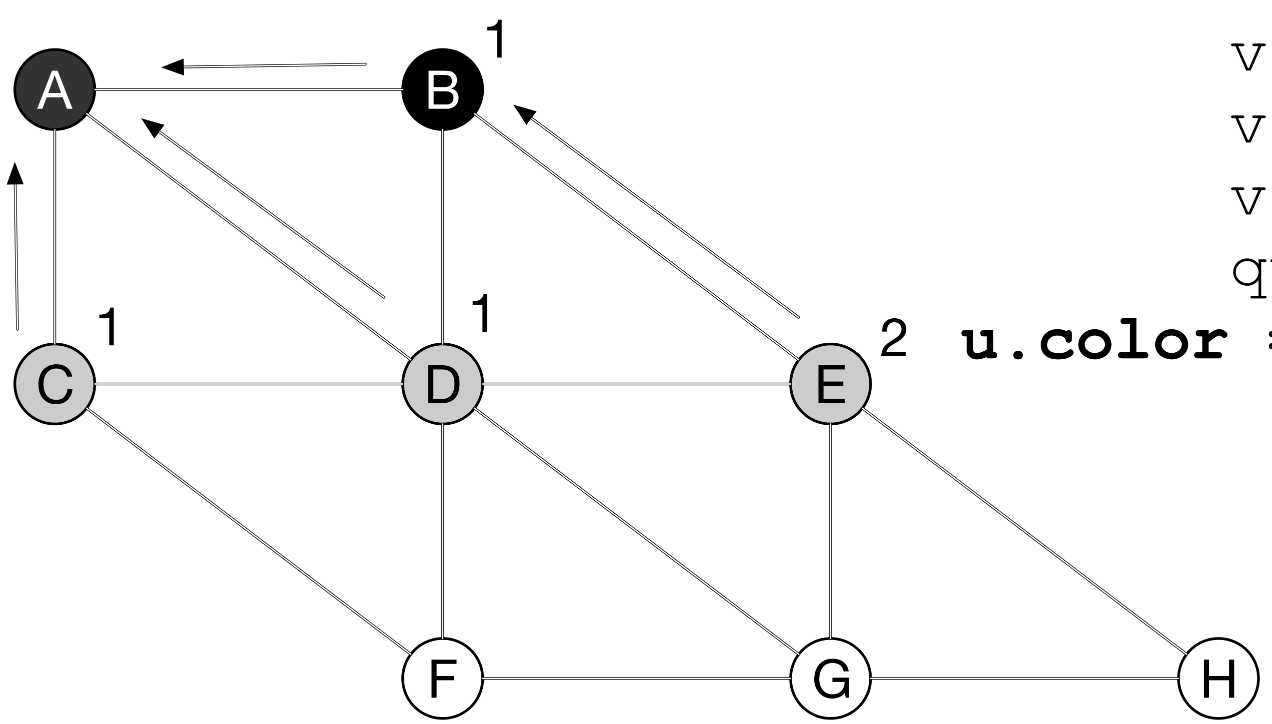


- queue = {C, D, E}

Breadth First Search

- queue = {C,D,E}
- u = B

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
           v.dist = u.dist+1
           v.pred = u
           queue.append(v)
    u.color = 'black'
```

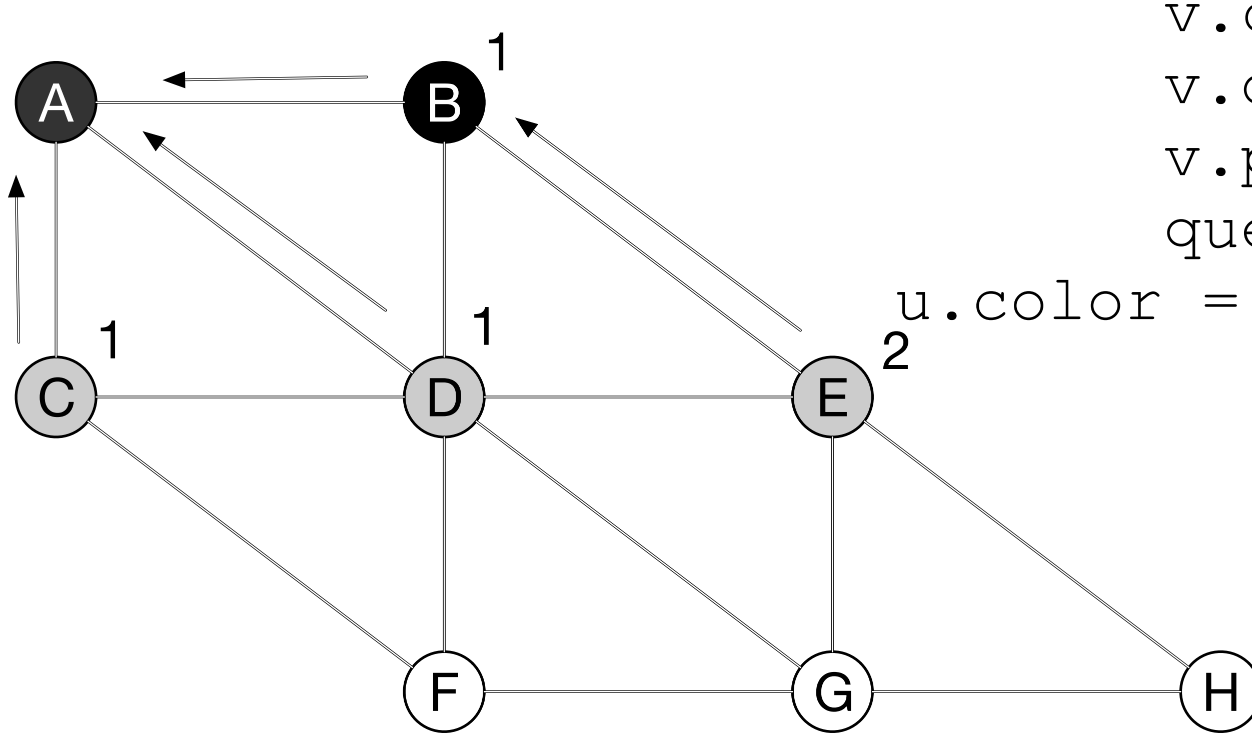


- queue = {C,D,E}

Breadth First Search

- queue = {C,D,E}
- u = C

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
        if v.color == 'white'
            v.color = 'gray'
            v.dist = u.dist+1
            v.pred = u
            queue.append(v)
    u.color = 'black'
```

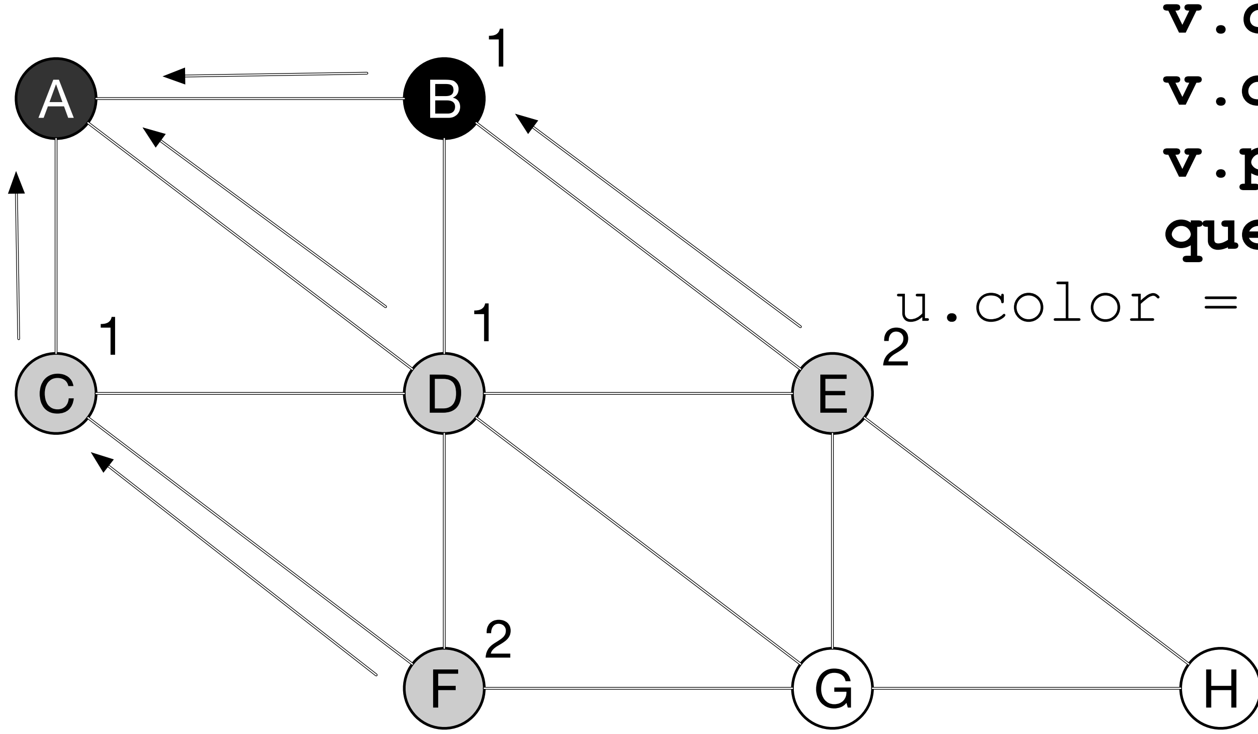


- queue = {D,E}

Breadth First Search

- queue = {D,E}
- u = C

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
           v.dist = u.dist+1
           v.pred = u
           queue.append(v)
    u.color = 'black'
```

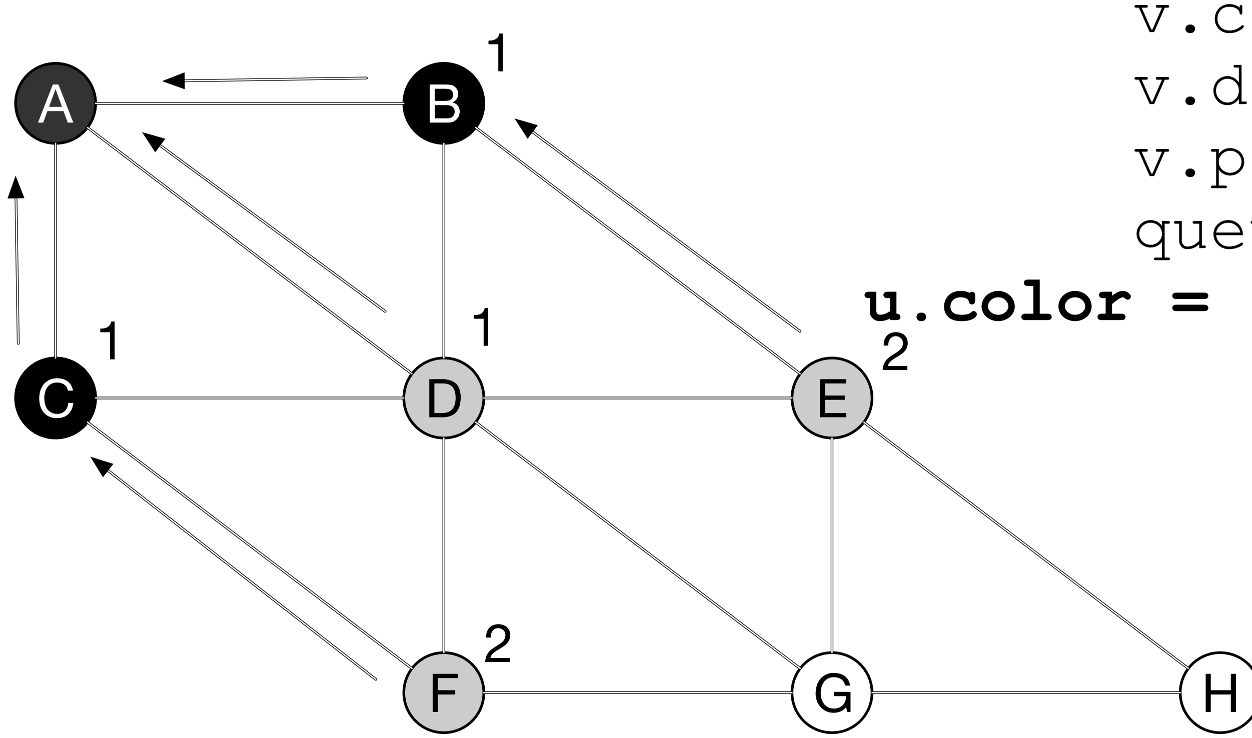


- queue = {D, E, F}

Breadth First Search

- queue = {D,E,F}
- u = C

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
           v.dist = u.dist+1
           v.pred = u
           queue.append(v)
    u.color = 'black'
```

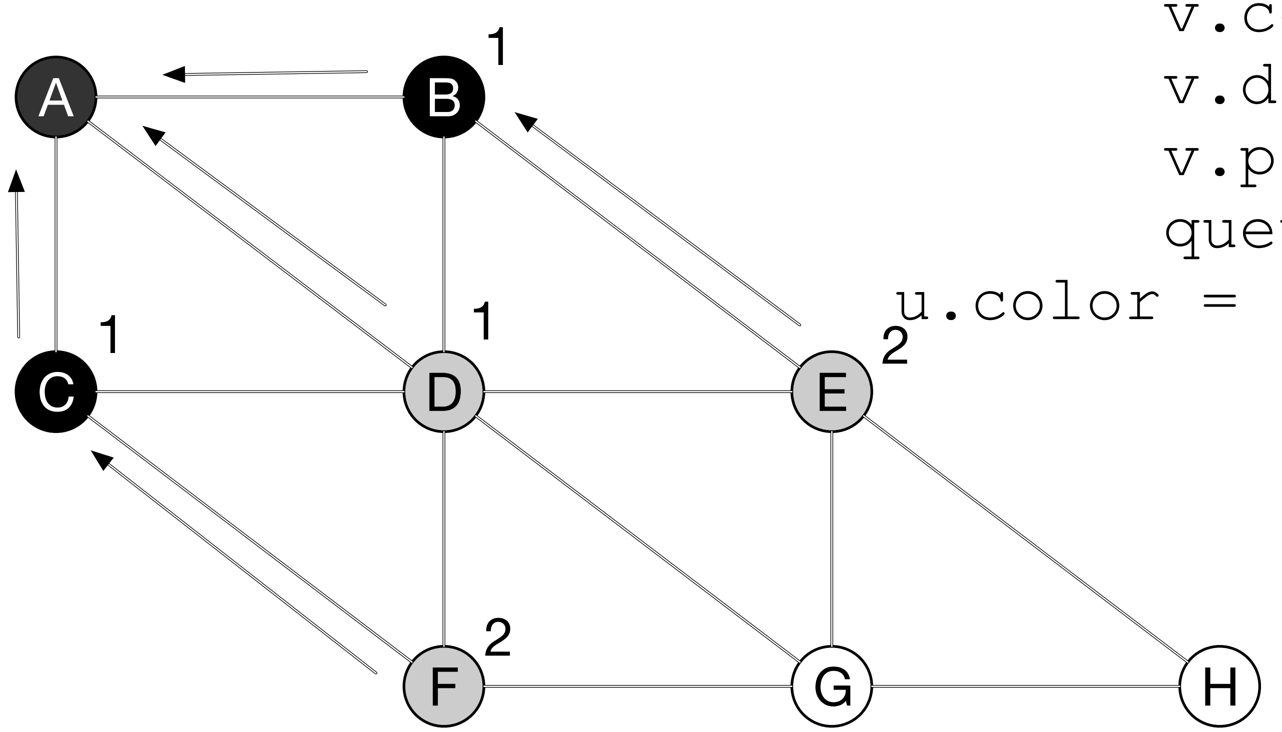


- queue = {D, E, F}

Breadth First Search

- queue = {D,E,F}
- u = D

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
        if v.color == 'white'
            v.color = 'gray'
            v.dist = u.dist+1
            v.pred = u
            queue.append(v)
    u.color = 'black'
```

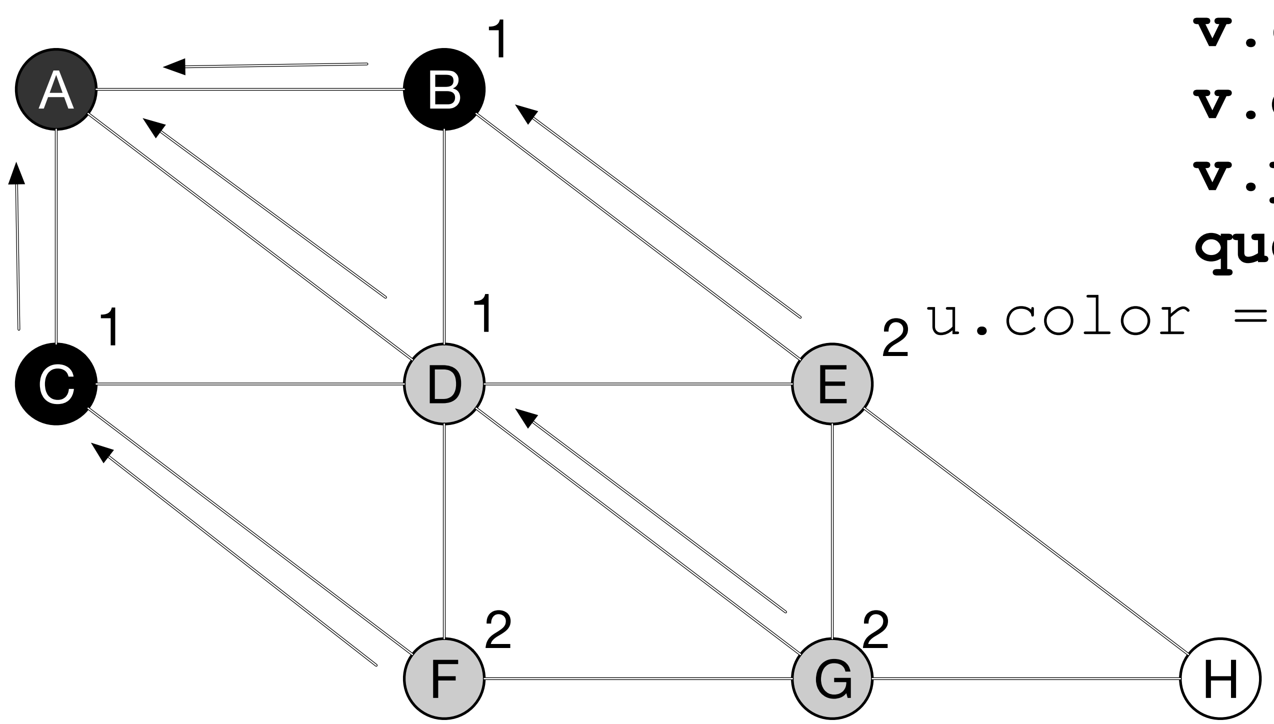


- queue = {E, F}

Breadth First Search

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
           v.dist = u.dist+1
           v.pred = u
           queue.append(v)
    u.color = 'black'
```

- queue = {E,F}
- u = D

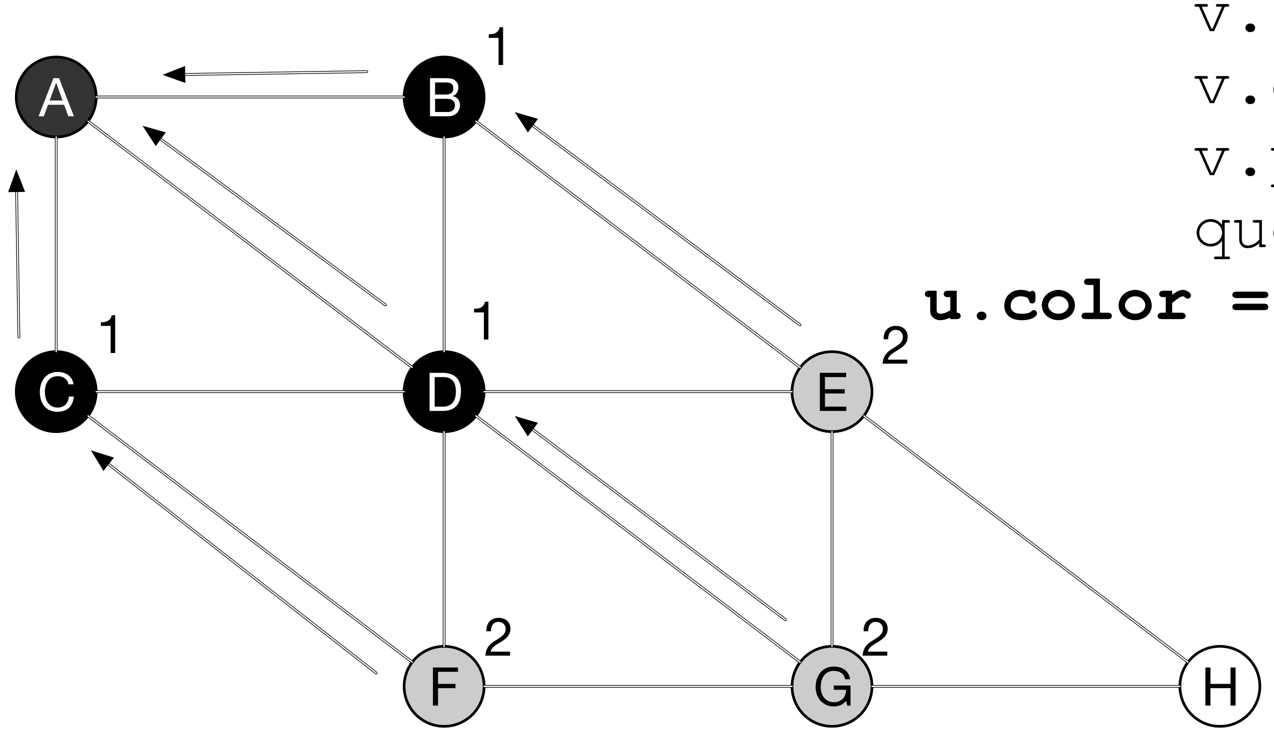


- queue = {E, F, G}

Breadth First Search

- queue = {E,F,G}
- u = D

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
           v.dist = u.dist+1
           v.pred = u
           queue.append(v)
    u.color = 'black'
```



- queue = {E, F, G}

Breadth First Search

- queue = {E,F,G}
- u = E

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
        if v.color == 'white'
            v.color = 'gray'
            v.dist = u.dist+1
            v.pred = u
            queue.append(v)
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```
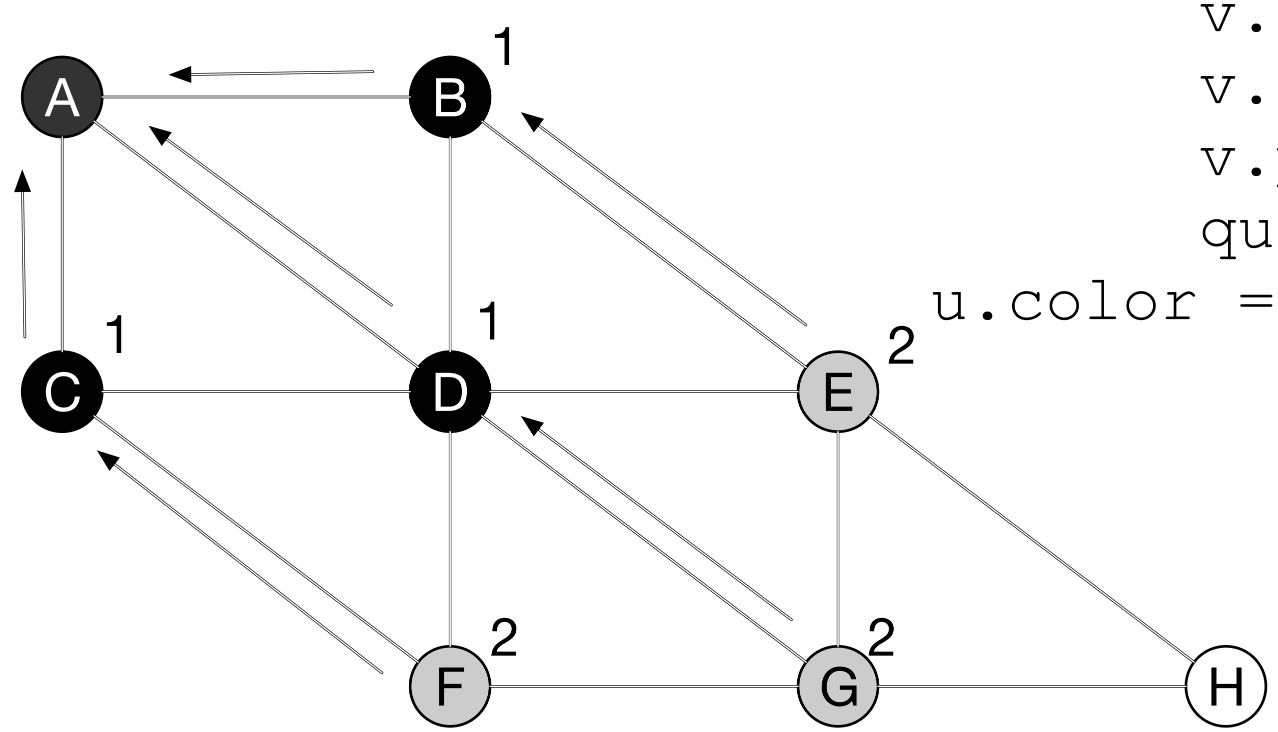
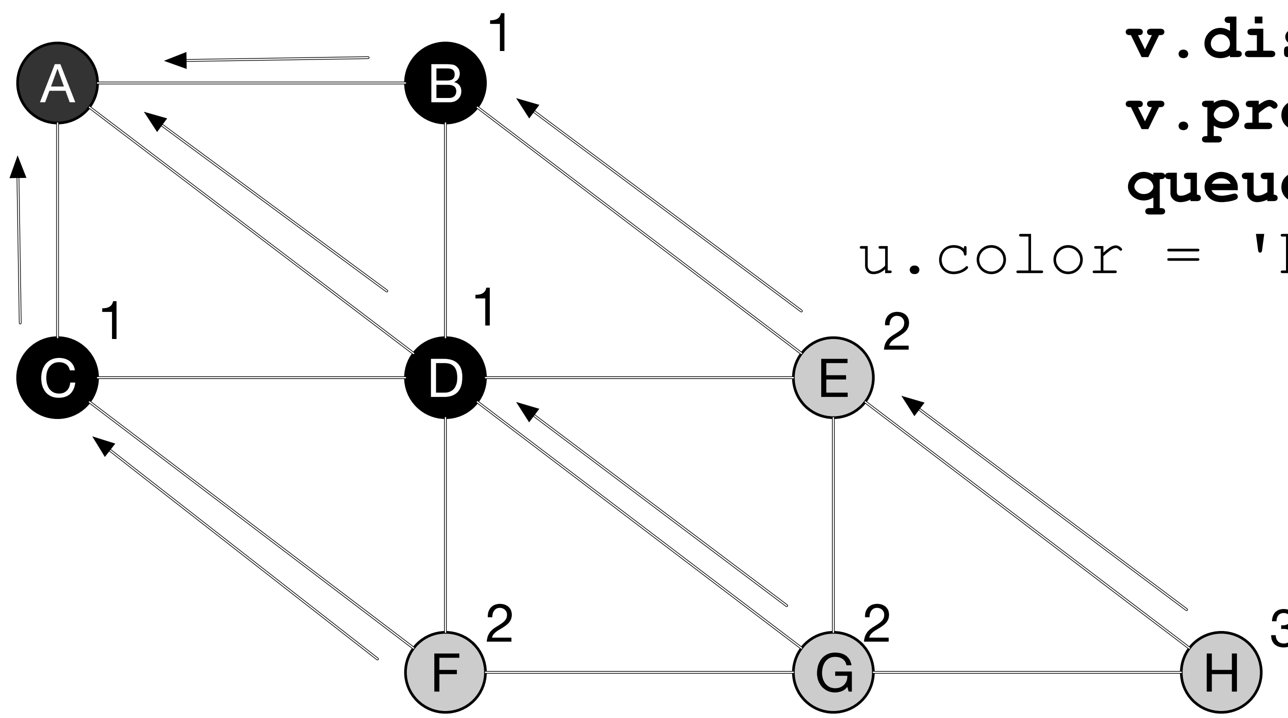



- queue = { F, G}

Breadth First Search

- queue = {F,G}
- u = E

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
        if v.color == 'white'
            v.color = 'gray'
            v.dist = u.dist+1
            v.pred = u
            queue.append(v)
    u.color = 'black'
```

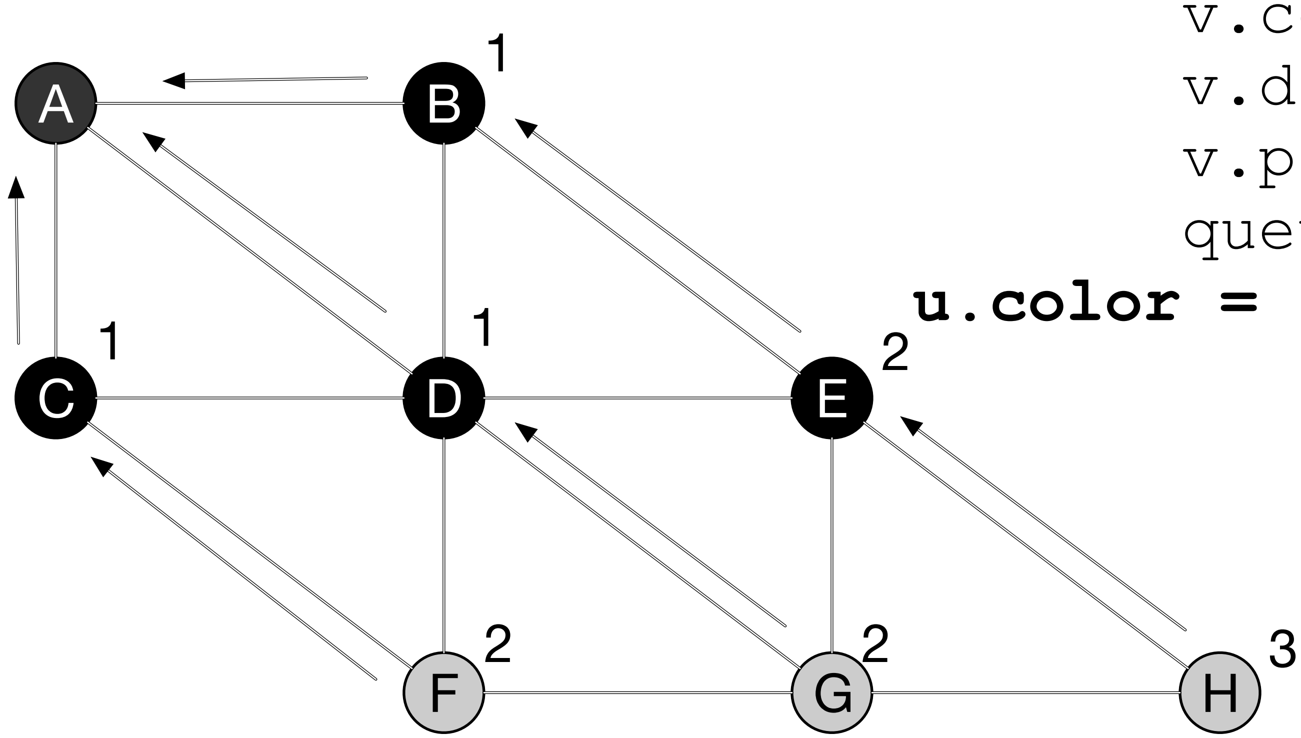


- queue = { F, G, H}

Breadth First Search

- queue = {F,G,H}
- u = E

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
           v.dist = u.dist+1
           v.pred = u
           queue.append(v)
    u.color = 'black'
```



- queue = { F, G, H}

Breadth First Search

- queue = {G,H}
- u = F

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
        if v.color == 'white'
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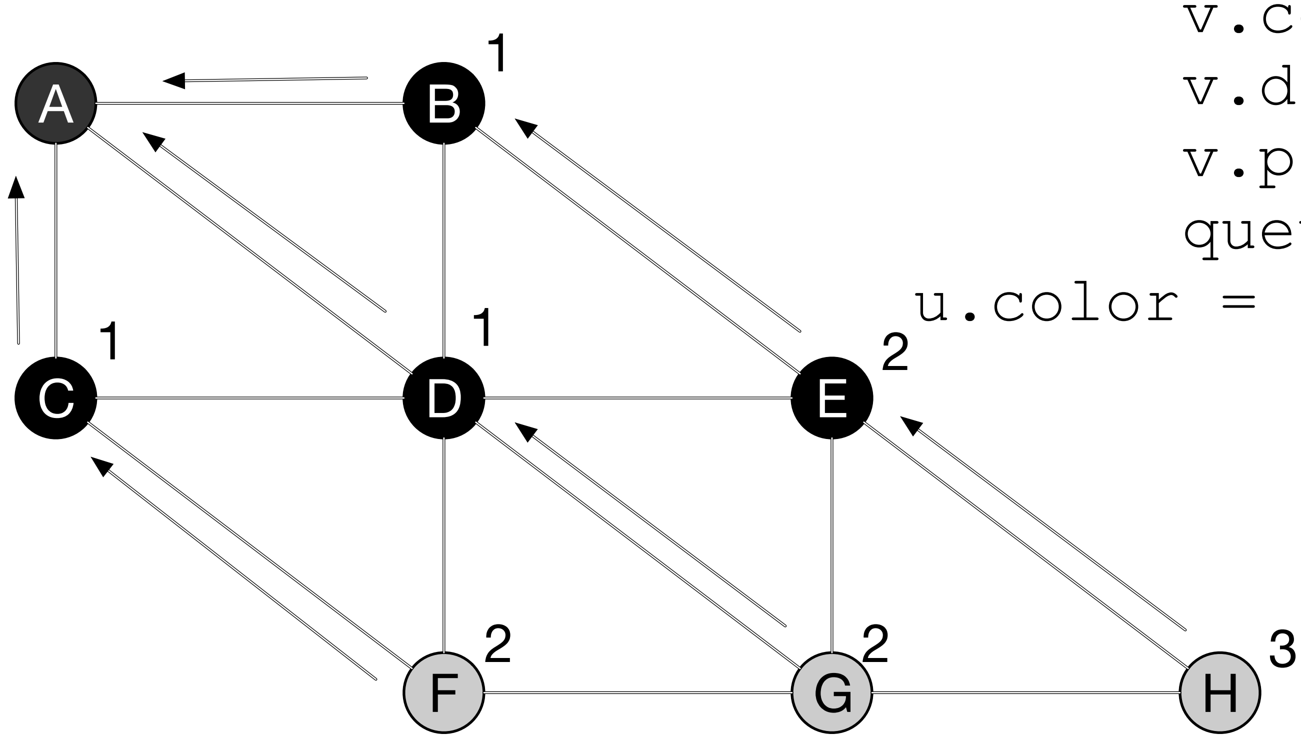

- queue = {G, H}

Breadth First Search

- queue = {G,H}
- u = F

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
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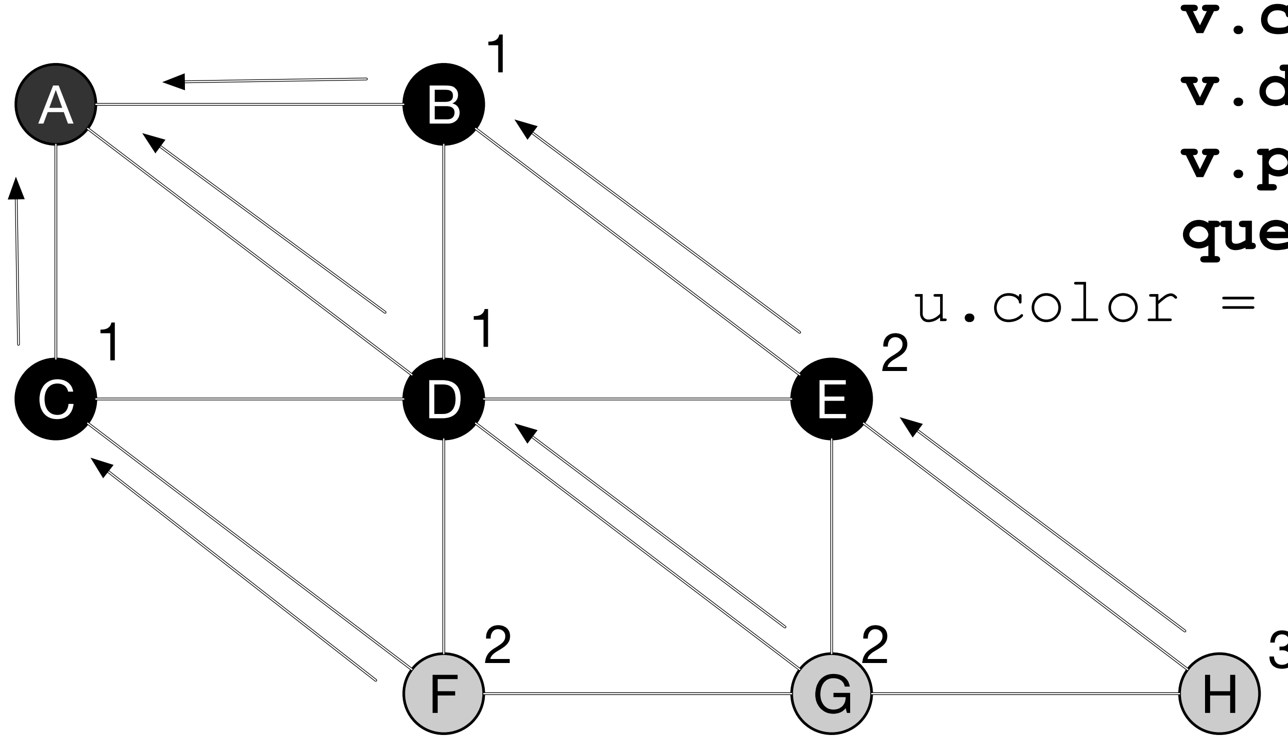

- queue = {G, H}

Breadth First Search

- queue = {G,H}
- u = F

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
           v.dist = u.dist+1
           v.pred = u
           queue.append(v)
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```
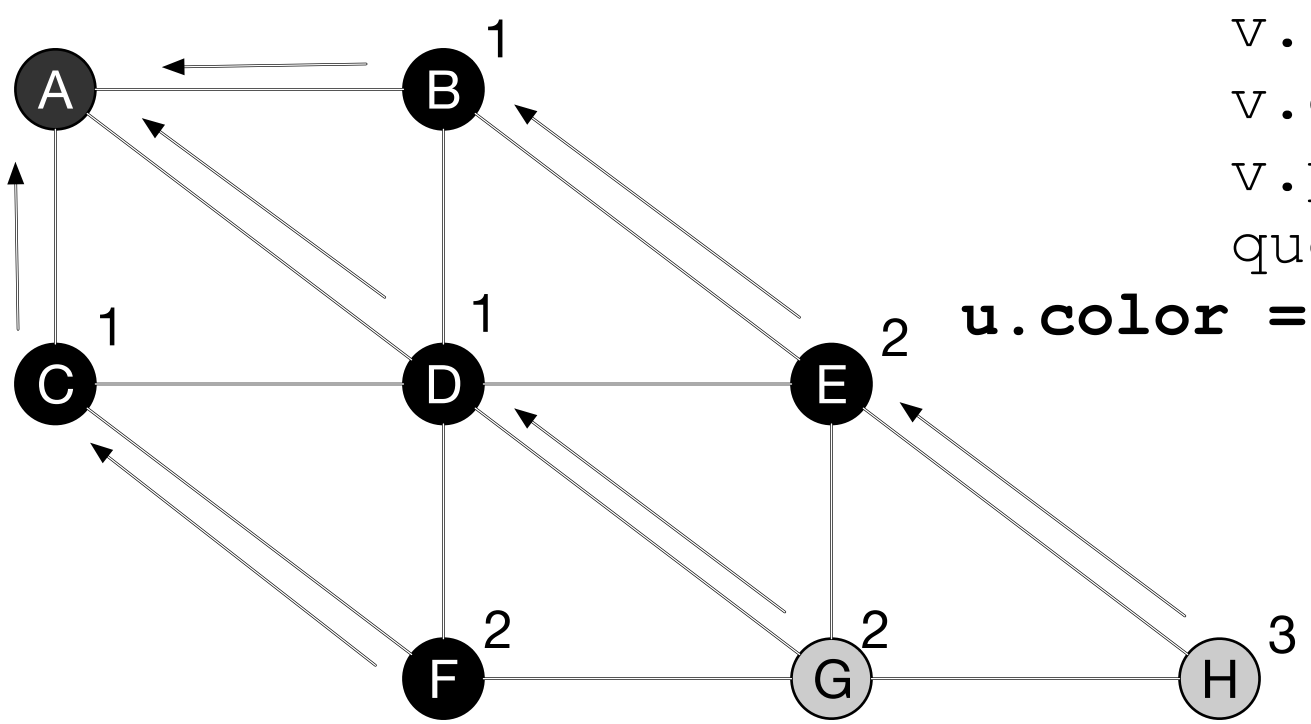
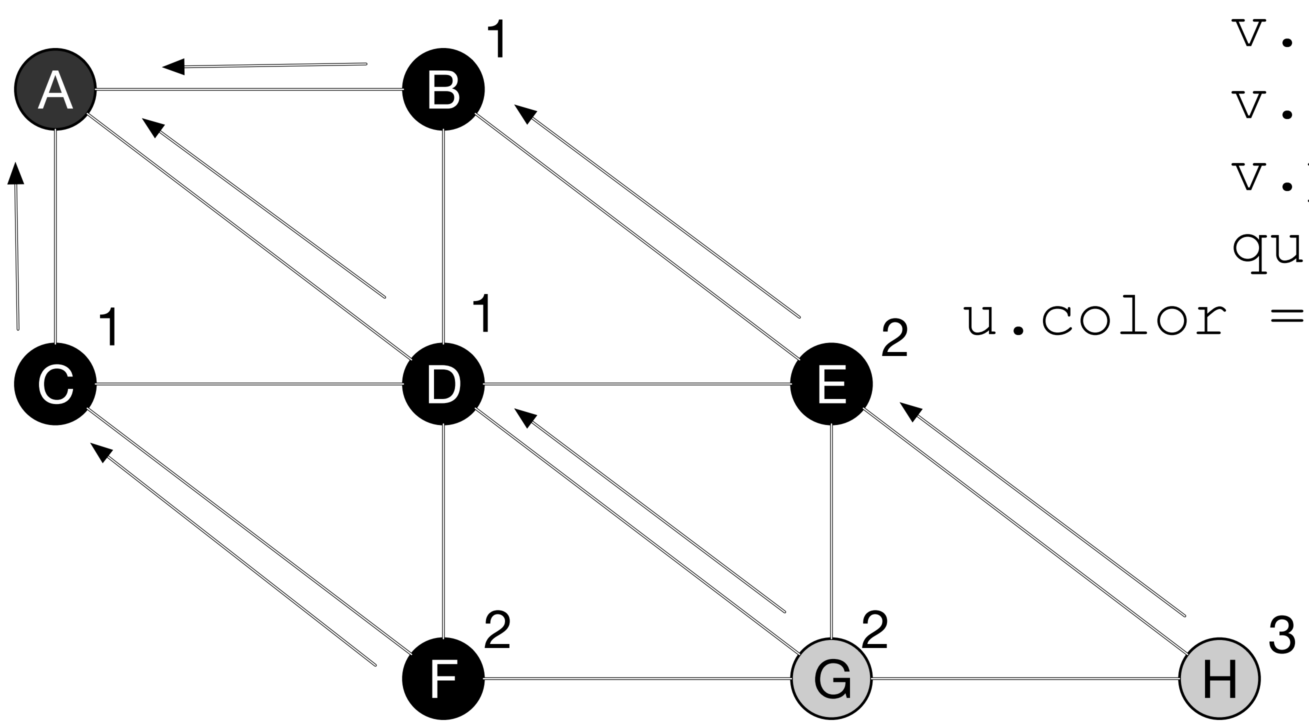



- queue = {G, H}

Breadth First Search

- queue = {G,H}
- u = G

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
           v.dist = u.dist+1
           v.pred = u
           queue.append(v)
    u.color = 'black'
```



- queue = {H}

Breadth First Search

- queue = {G,H}
- u = G

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
           v.dist = u.dist+1
           v.pred = u
           queue.append(v)
    u.color = 'black'
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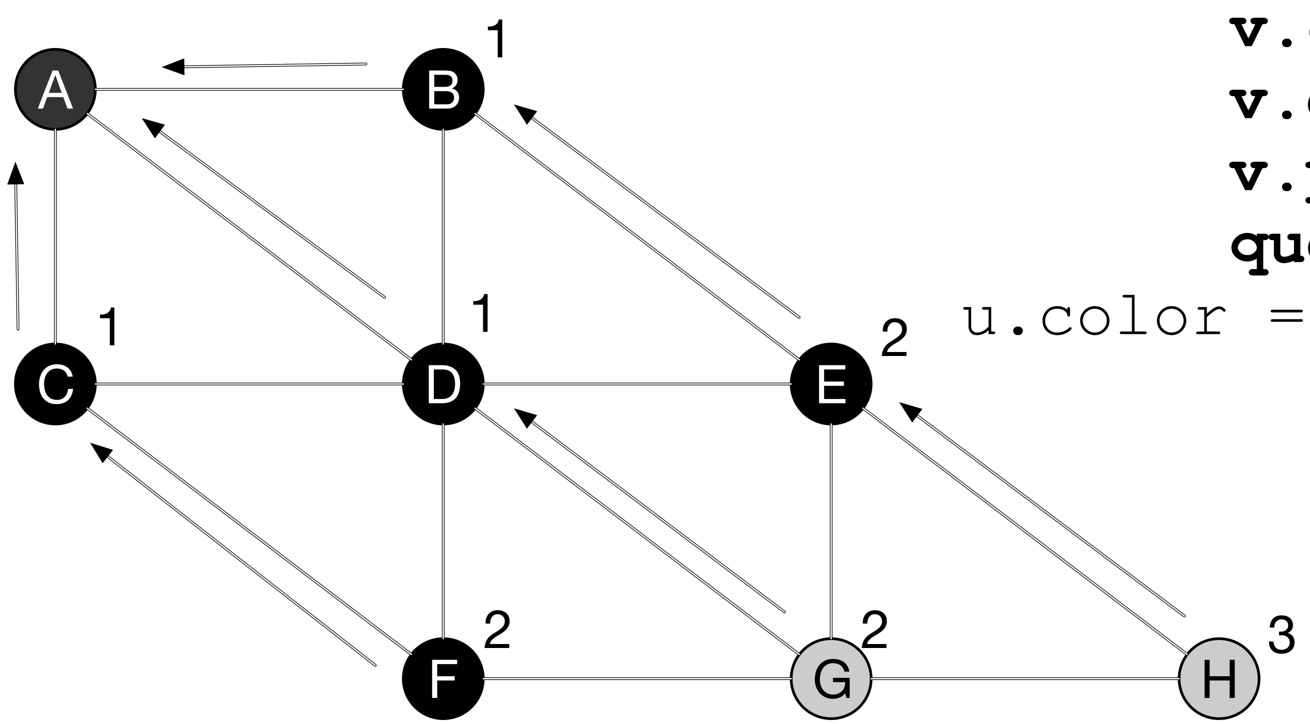

- queue = {H}

Breadth First Search

- queue = {H}
- u = G

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
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           v.pred = u
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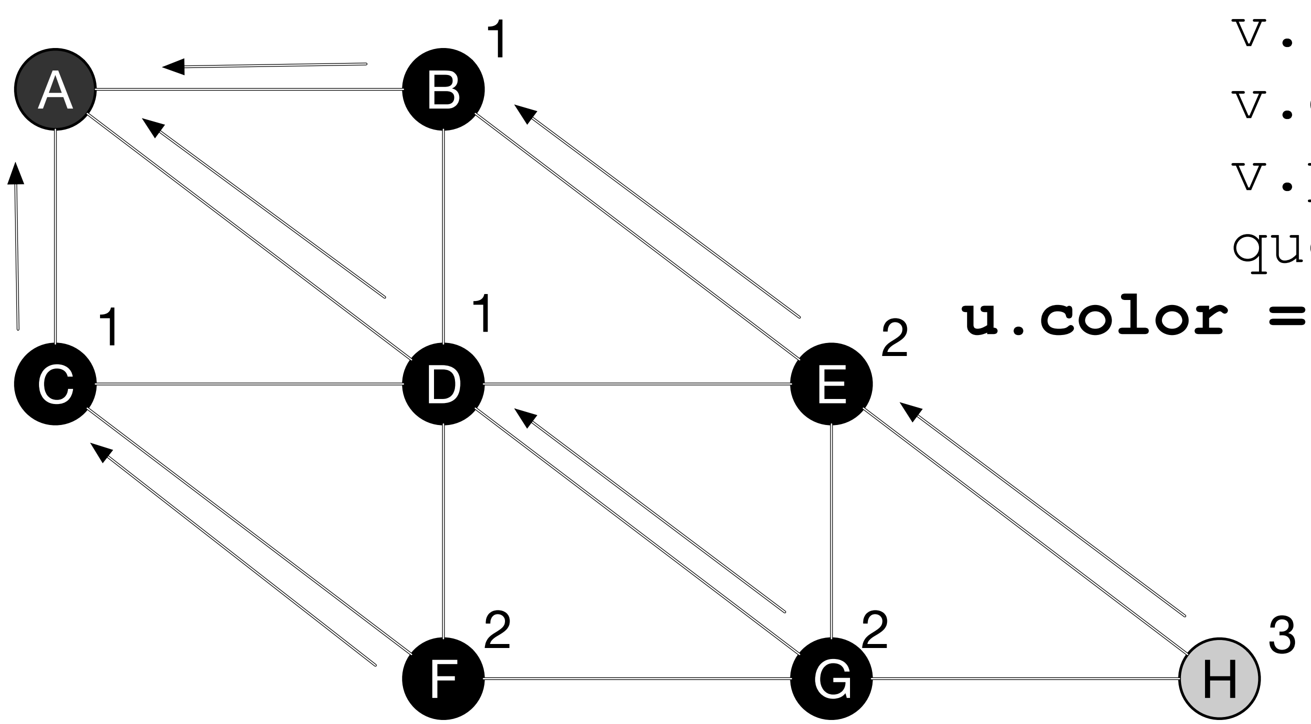

- queue = {H}

Breadth First Search

- queue = { }
- u = H

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
           v.dist = u.dist+1
           v.pred = u
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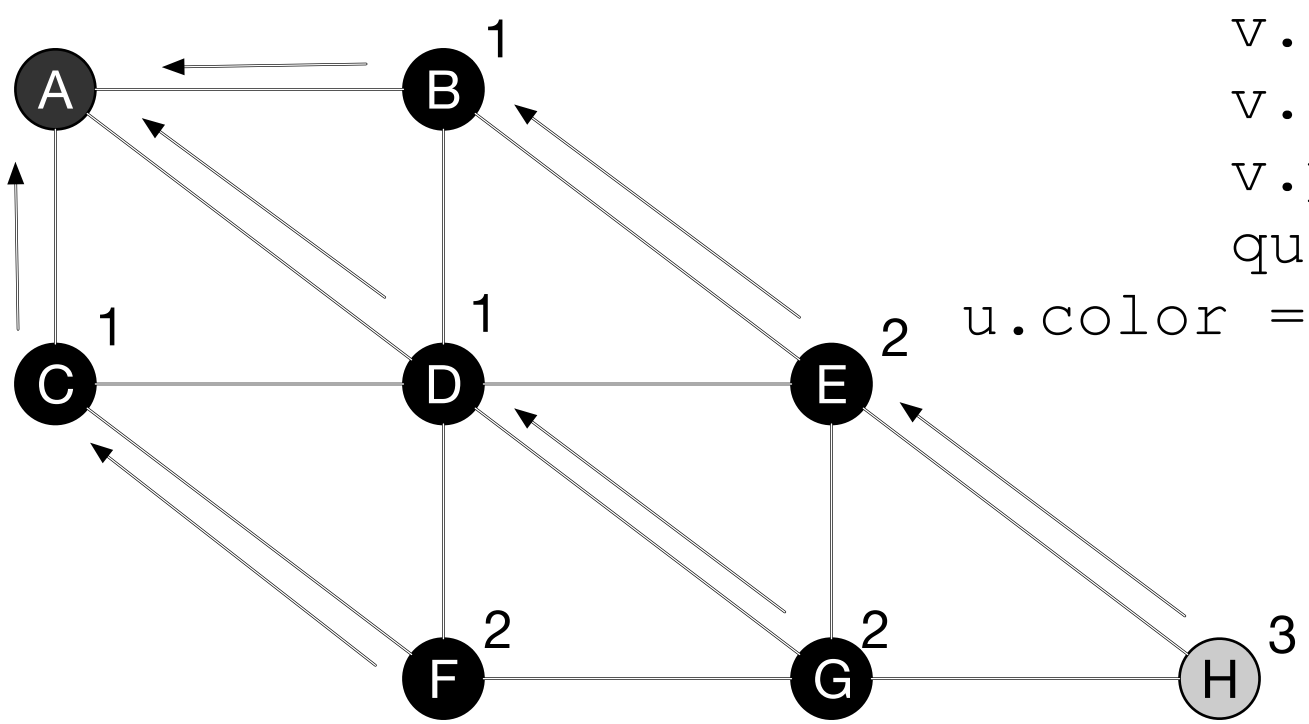

- queue = { }

Breadth First Search

- queue = { }
- u = H

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
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           v.pred = u
           queue.append(v)
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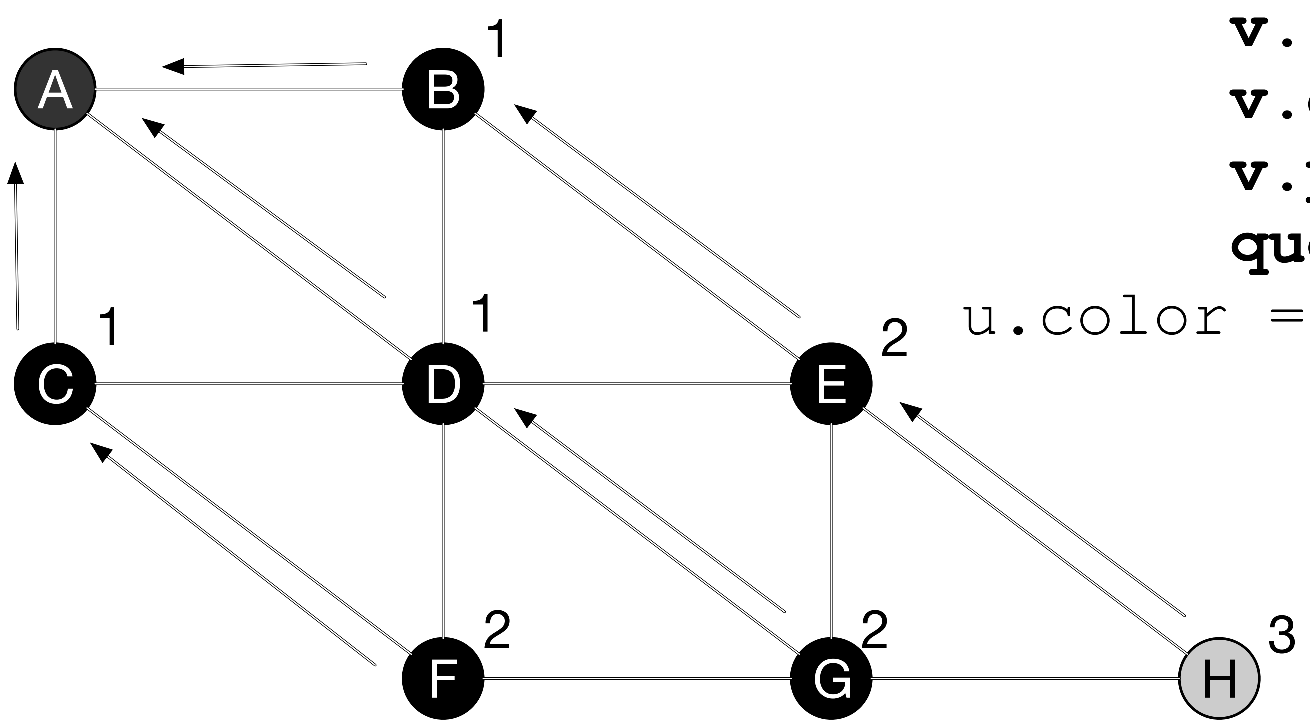
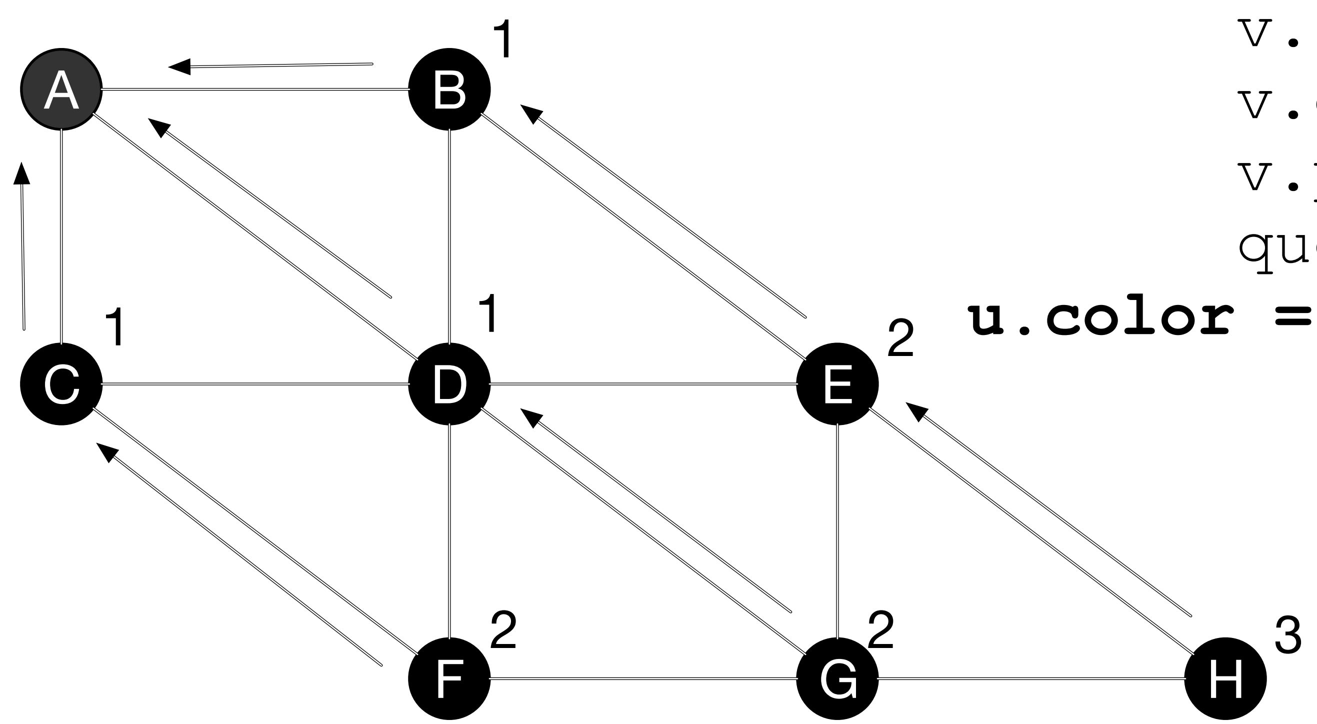


- queue = { }

Breadth First Search

- queue = { }
- u = H

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
           v.dist = u.dist+1
           v.pred = u
           queue.append(v)
    u.color = 'black'
```



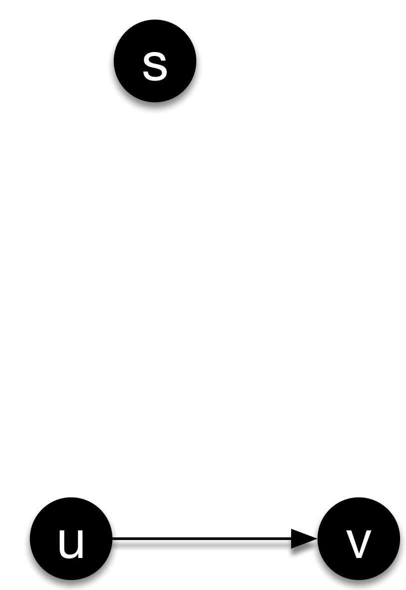
- queue = { }

Breadth First Search

- As you can see, BFS is just a version of Dijkstra's algorithm
- Distance calculates accurately the distance from the starting point
- The pred property allows us to generate a shortest path from the initial node
- We now prove these properties exactly

Breadth First Search

- Lemma: Let $G = (E, V)$ be an undirected or directed graph. Let $s \in V$ be an arbitrary vertex. Then for any edge $(u, v) \in E$
  - $\delta(s, v) \leq \delta(s, u) + 1$



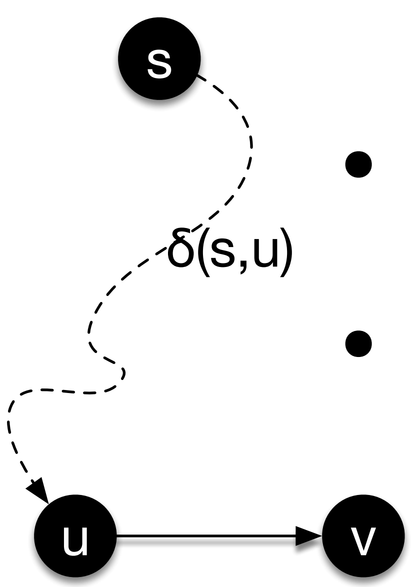
- Recall: $\delta(a, b)$ is the length of a shortest path from a to b

Breadth First Search

- Proof:
  - Assume first that $\delta(s, u) = \infty$, i.e. there is no path from s to u
    - Then $\delta(s, v) \leq \infty = \delta(s, u) + 1$ regardless whether there is a path from s to v.

Breadth First Search

- Proof:
  - Next assume that $\delta(s, u) < \infty$, i.e. that there is a path from s to u.
    - Extend this path to a path from s to v.
    - This path has length $\delta(s, u) + 1$.
    - Then $\delta(s, v) = \min($Lenght of a path from s to $v)$
      - $\leq$ Length of this path
      - $= \delta(s, u) + 1$



Breadth First Search

- Lemma: Let Let $G = (E, V)$ be an undirected or directed graph. Let $s \in V$ be an arbitrary vertex. Run BFS on G and s. Then for every vertex $v \in V$, $v\,.\,\text{dist} \geq \delta(s, v)$.
  - This means that the calculated distance in BFS is at least as large as the actual distance

Breadth First Search

- Proof by induction on the number of enqueue operations
  - Notice that v.dist is assigned just when we are about to enqueue it

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
- - >      v.dist = u.dist+1
           v.pred = u
           queue.append(v)
    u.color = 'black'
```

Breadth First Search

- Induction Start:
  - When s is enqueued all distance properties are infinity
    - with the exception of s which has dist 0
  - At this point, for every vertex $v \in V$, $v\,.\,\text{dist} \geq \delta(s, v)$

Breadth First Search

- Induction step:
  - The value of the distance property only changes when we make the assignment just before enqueuing a white vector
  - Induction hypothesis implies $u\,.\,\text{dist} \geq \delta(s, u)$
  - Therefore $v\,.\,\text{dist} = u\,.\,\text{dist}+1 \geq \delta(s, u) + 1 \geq \delta(s, v)$

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
- - >      v.dist = u.dist+1
           v.pred = u
           queue.append(v)
    u.color = 'black'
```

Breadth First Search

- Afterwards, the vertex v is no longer white and never changes its distance value

Breadth First Search

- We now need to see more closely how the algorithm works:
  - We can think of the queue as the boundary between black and white vertices that moves slowly away from s
- Lemma: If the queue has vertices $(v_1, v_2, \ldots, v_n)$ with v_1 being the head, then
  - v_n . dist $\leq v_1$. dist+1
    - and
  - v_i . dist $\leq v_{i+1}$. dist for $i = 1,2,\ldots,n-1$

Breadth First Search

- Proof by induction on the number of queue operations
  - Initially, the queue has only s in it, so the property certainly holds
  - The queue changes through enqueuing and dequeuing operations

Breadth First Search

- If the head v_1 is dequeued, v_2 becomes the new head.
  - (If there is no v_2 then the queue is empty, and the assertion holds vacuously)
  - Before dequeuing, v_1 . dist $\leq v_2$. dist, therefore v_n . dist $\leq v_1$. dist$+1 \leq v_2$. dist$+1$
  - Therefore, the first inequality is true
  - The second assertion just looses the first inequality

Breadth First Search

- If a new element v_{n+1} is enqueued, we just dequeued a vertex u and are adding all white vertices adjacent to u

```
while queue:
    u = queue.pop(0)
    for v in u.adjacency:
       if v.color == 'white'
           v.color = 'gray'
- - >      v.dist = u.dist+1
           v.pred = u
           queue.append(v)
    u.color = 'black'
```

- Therefore, $v_{n+1}.\text{dist} = u.\text{dist}+1$.
- By induction hypothesis, $u.\text{dist} \leq v_1.\text{dist}$ because u and v_1 were just in the same queue

Breadth First Search

- Therefore
  - $v_{r+1}\,.\,\text{dist} = u\,.\,\text{dist}{+}1 \leq v_1\,.\,\text{dist}{+}1$
- Proving the first assertion

Breadth First Search

- From the induction hypothesis, we also have
  - v_n . dist $\leq u$. dist$+1$
- which implies that
  - v_n . dist $\leq u$. dist$+1 \leq v_{n+1}$. dist
- This is the only new part of the second assertion

Depth First Search

- Breadth first search uses a queue
  - In Python, a queue is a list to which you append and from which you pop
  - C++ and Java have libraries that implement queues

```
def bfs(G, s):
    for u in G.Vertices:
        u.color = "white"
        u.d = infty
        u.pred = Null
    s.color="gray"
    s.d = 0
    s.pred = Null
    queue = Queue.queue()
    queue.enqueue(s)
    while queue:
        u = queue.head()
        for v in u.adjacency_list:
            if v.color=="white"
                v.color = "gray"
                v.d = u.d + 1
                v.pred = u
                queue.enqueue(v)
        u.color="black"
```

Depth First Search

- Depth first search replaces the queue with a stack
  - This changes the behavior of the algorithm considerably
  - Remarkably, the resulting Depth First Search is the more important and interesting algorithm

Depth First Search

- Depth first search
- Version 1
    - Change queue into stack
    - Get rid of the distance

```
def dfs(G, s):
    for u in G.Vertices:
        u.color = “white”
        u.d = infty
        u.pred = Null
    s.color=“gray”
    s.pred = Null
    queue = Stack.stack()
    stack.push(s)
    while stack:
        u = stack.pop()
        for v in u.adjacency_list:
            if v.color==“white”
                v.color = “gray”
                v.pred = u
                stack.push(v)
        u.color=“black”
```

Depth First Search

- We add visiting times to our nodes:
  - Discovered time
    - When a node turns gray
  - Finished time
    - When a node turns black
- Because
  - some derived algorithms use it
  - in order to argue about DFS
- Whenever we change a node color, we increment a clock

Depth First Search

- Unlike BFS, a typical DFS will want to classify all nodes
    - Have a DFS_Visit(start_node) that starts in a node and visit what can be available
    - Have a DFS( ) function that uses the visit function repeatedly if necessary

Depth First Search

```
dfs_visit(u):
    global clock
    clock += 1
    u.d = clock
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            v.pred = u
            dfs_visit(v)
    u.color = 'black'
    clock += 1
    u.f = clock
```

Depth First Search

```
dfs(G):
   for vertex in G.V:
      vertex.color = 'white'
      vertex.pred = None
   global clock = 0
   for vertex in G.V:
       if vertex.color == 'white':
          dfs_visit(vertex)
```

Depth First Search

- Understanding the algorithm
  - The stack is hidden in the recursive call
  - We can unroll it
  - But need to be careful as something on the stack can be already found and processed via another route

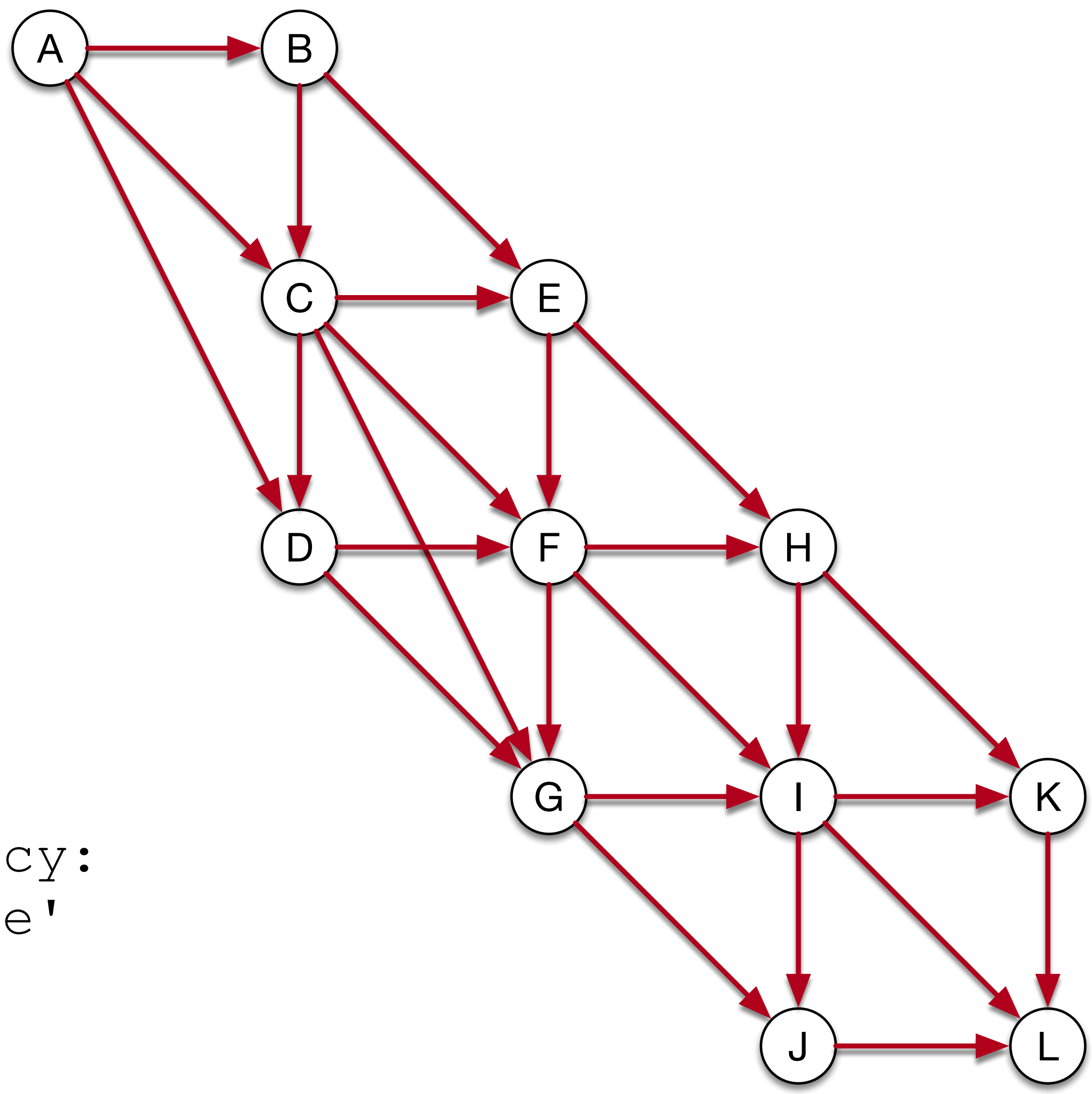
Depth First Search

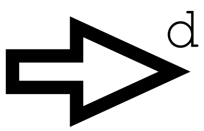
```
dfs_visit(u):
   stack = [u]
   while stack:
      u = stack.pop()
      if u.color=='white'
      for v in u.adjacency:
         if v.color=='white':
            stack.push(v)
```

Depth First Search

Start with C

OS stack is
    dfs_visit(C)





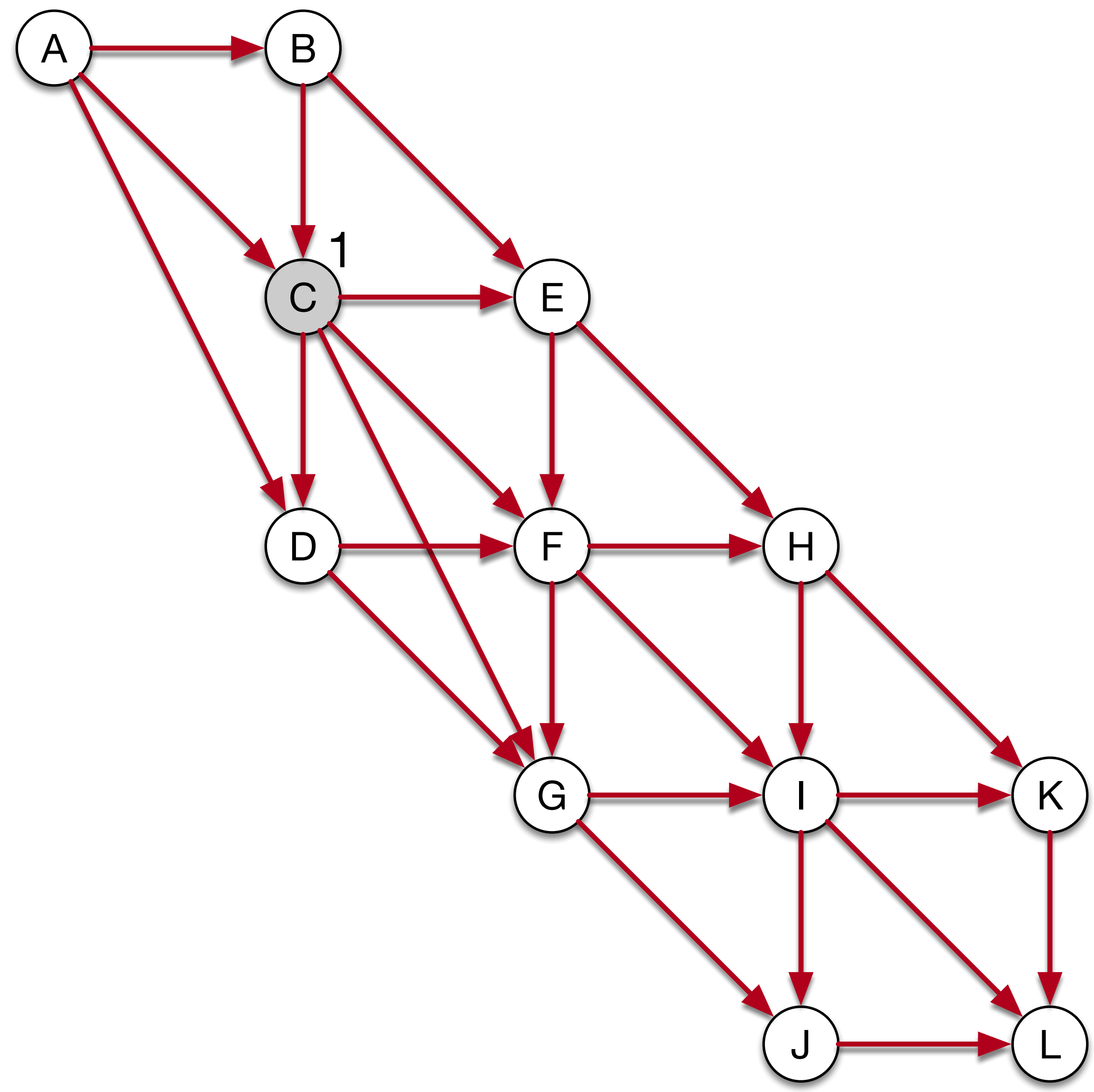
```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

Depth First Search

Start with C

OS stack
    dfs_visit(C)

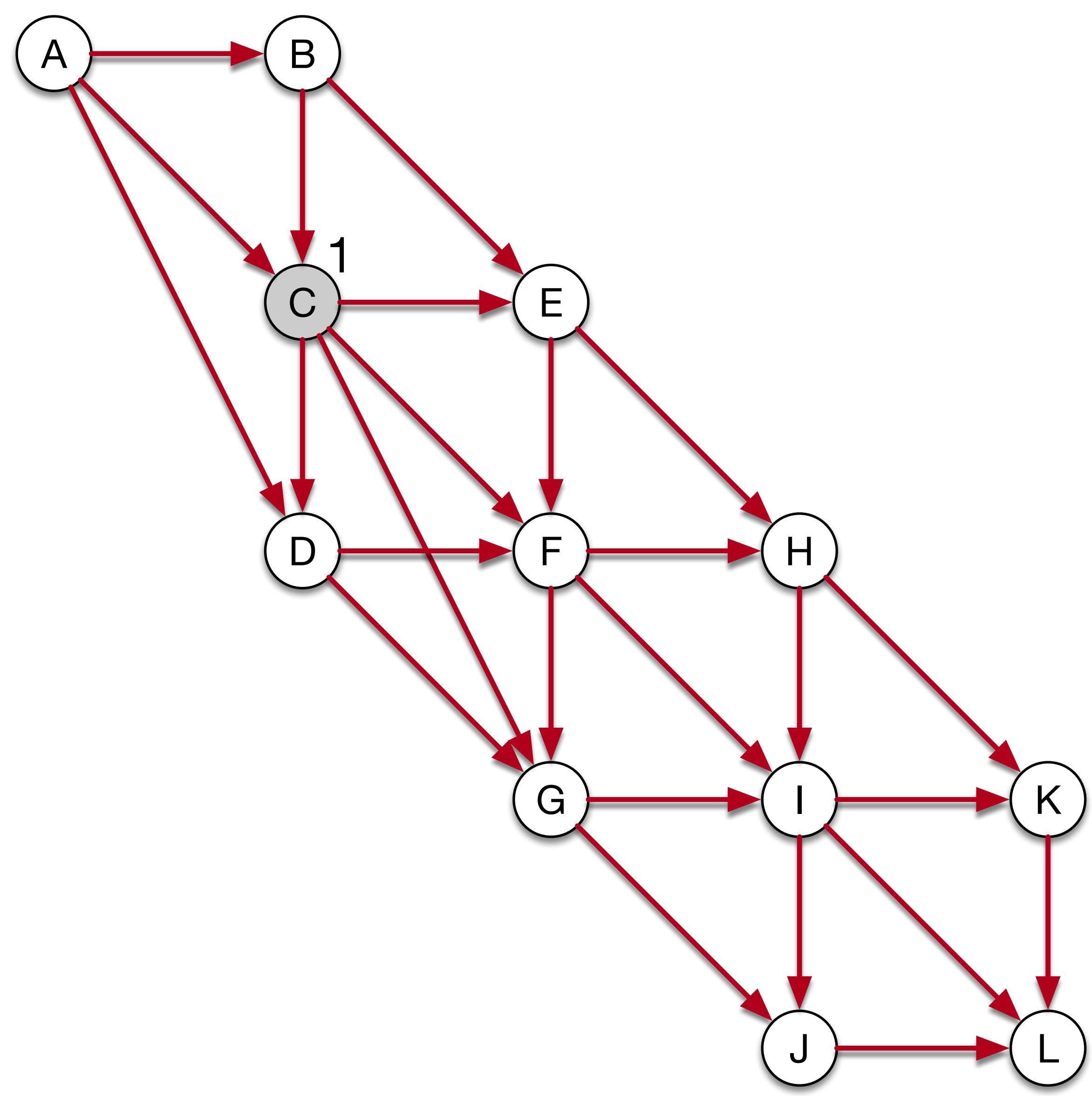
```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```



We set the clock to 1
We pick arbitrarily E
from the adjacency list

Depth First Search

```
OS stack
    dfs_visit(E)
    dfs_visit(C)
```

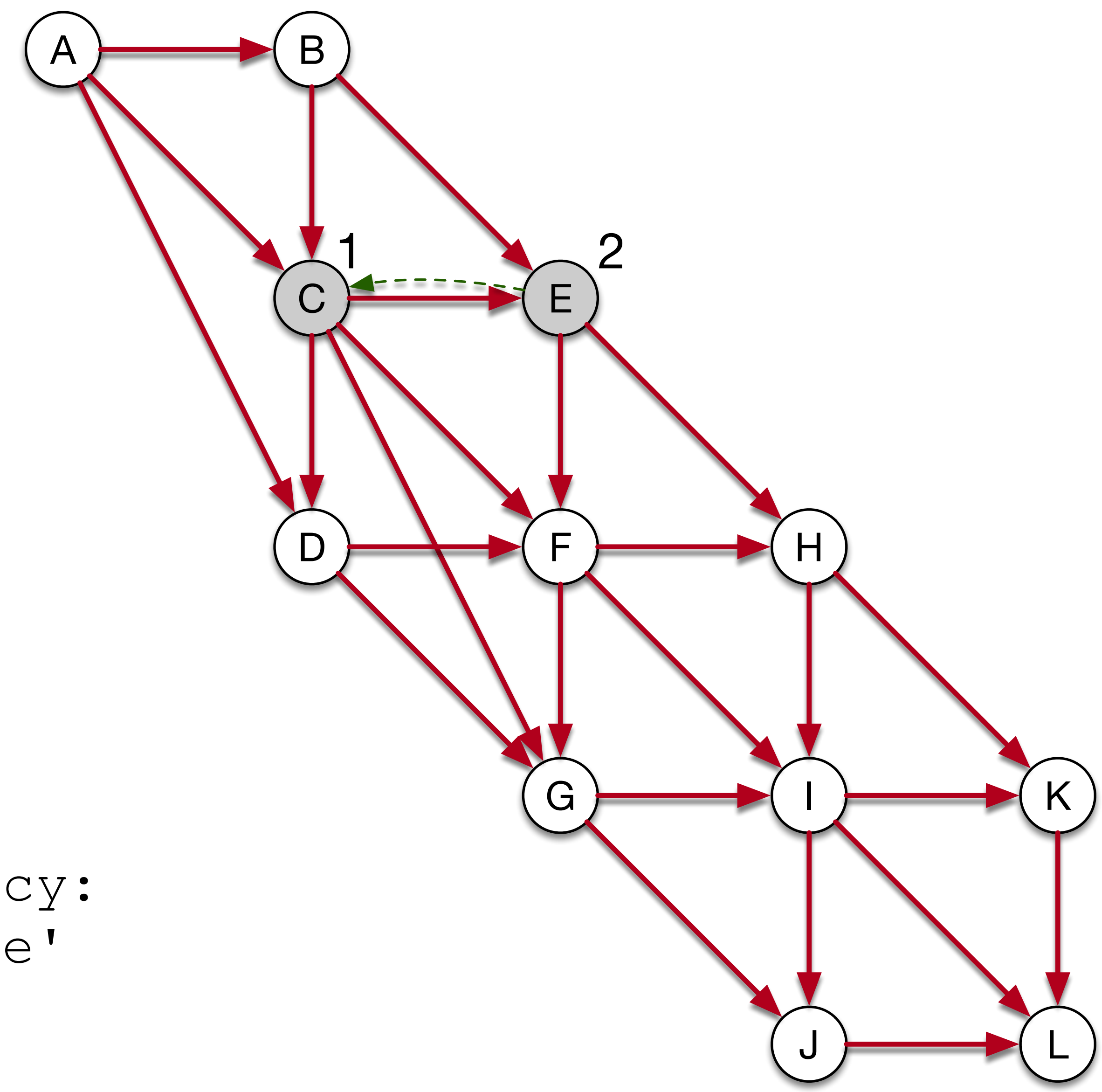



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

We call dfs_visit(E)

Depth First Search

```
OS stack
   dfs_visit(E)
   dfs_visit(C)
```

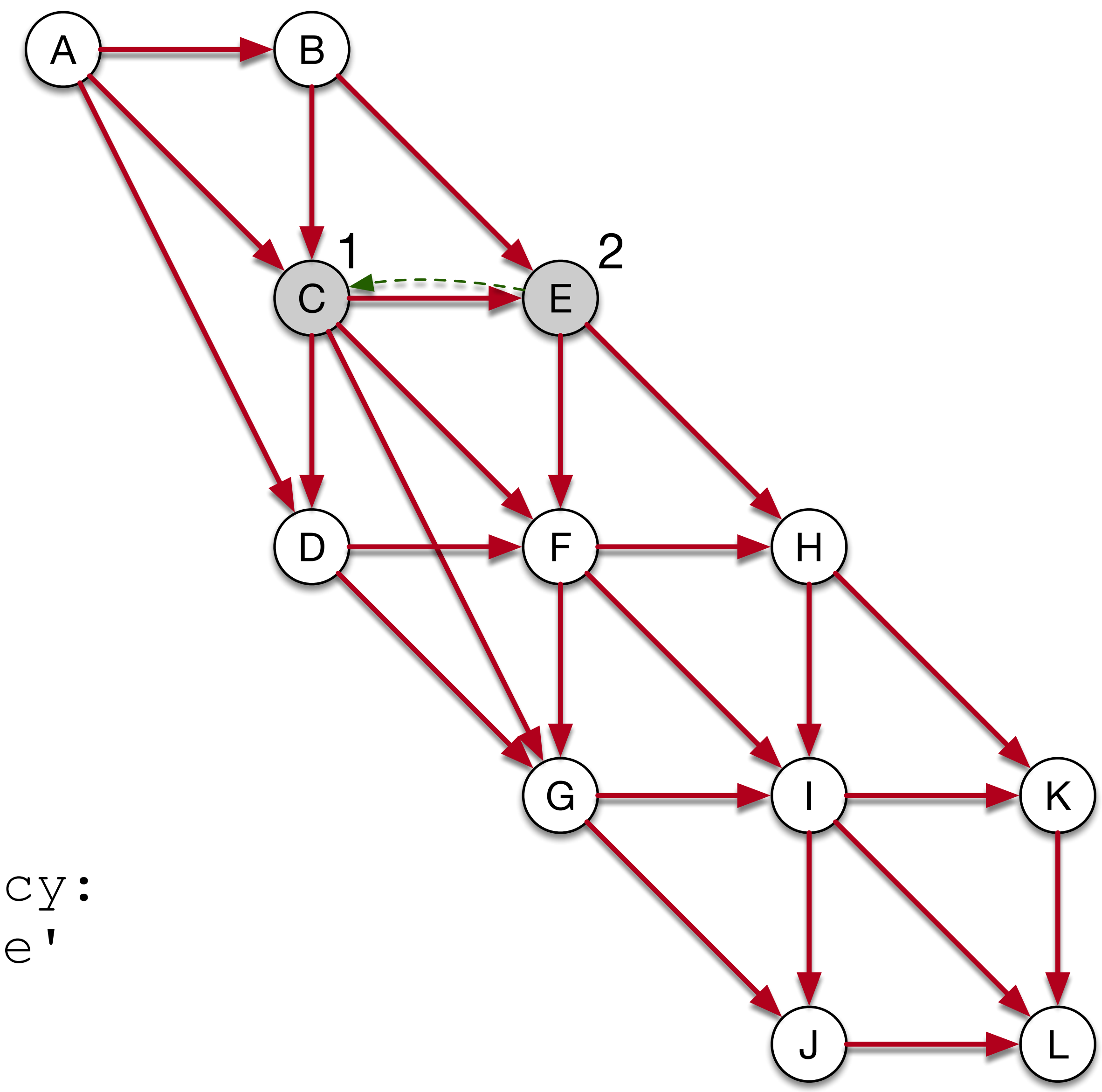



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

u = E, set E's discovery time and set the predecessor link

Depth First Search

```
OS stack
    dfs_visit(E)
    dfs_visit(C)
```

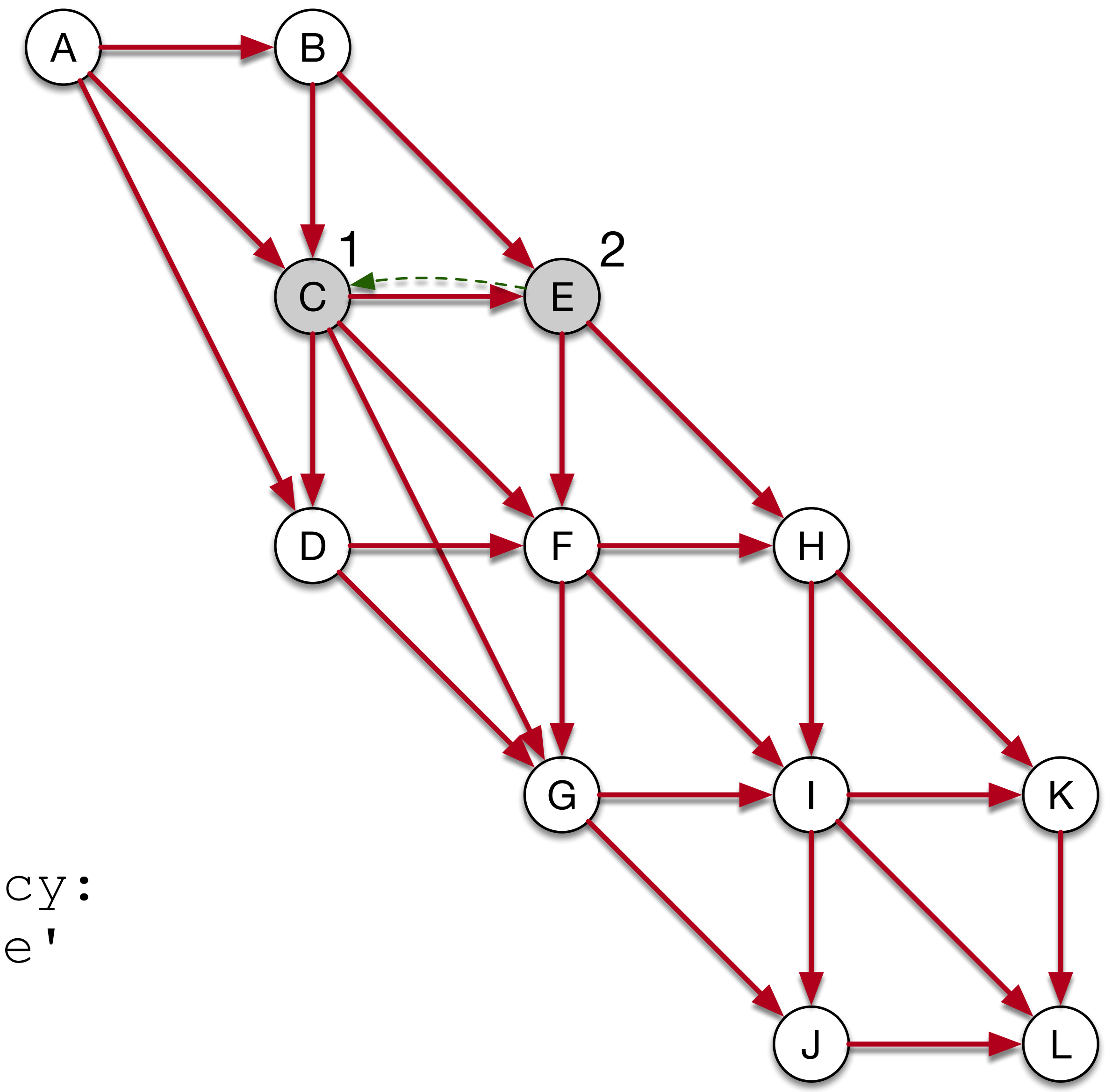



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

we pick v = H

Depth First Search

```
OS stack
    dfs_visit(E)
    dfs_visit(C)
```



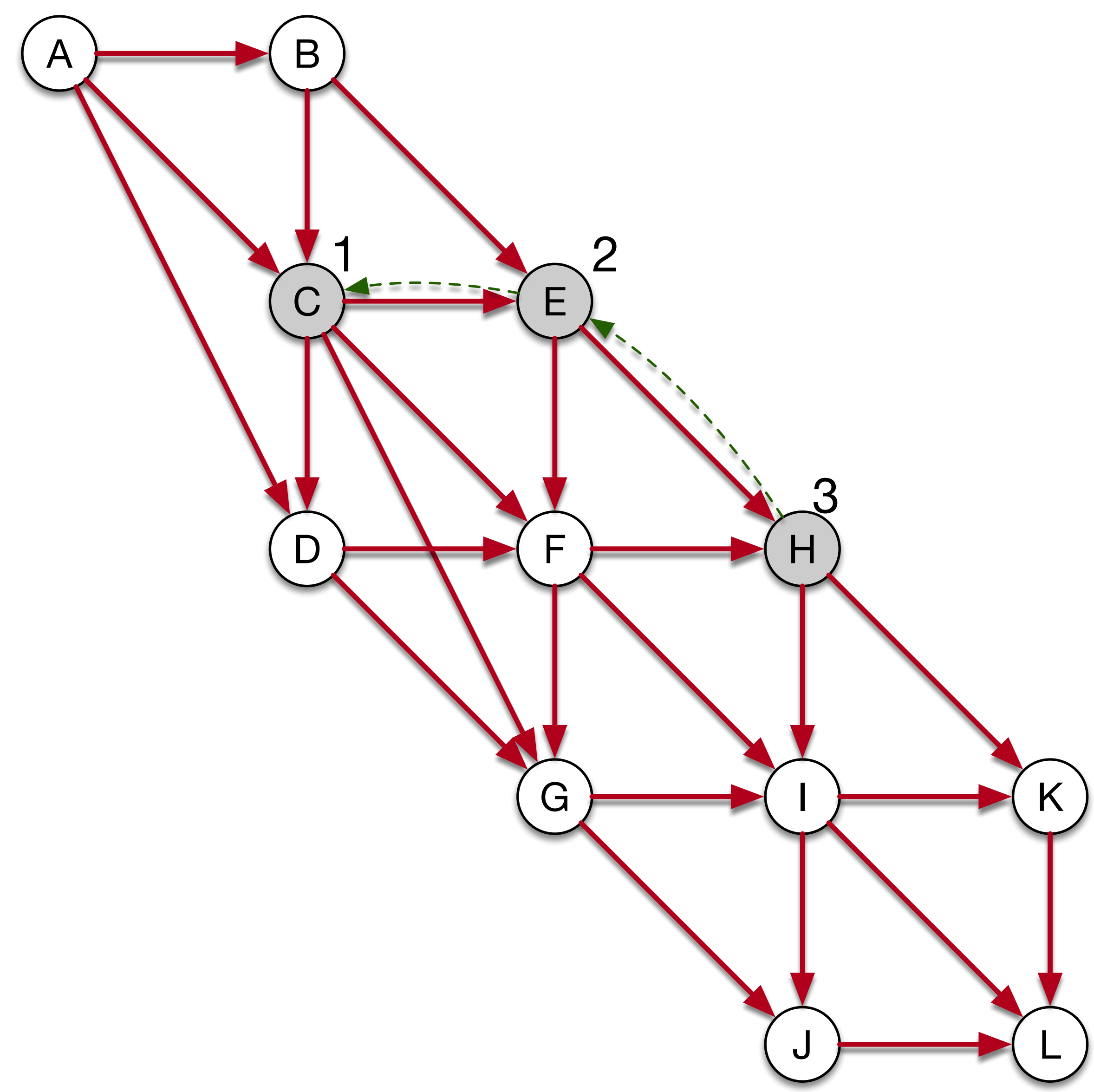

```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

we pick v = H and call dfs_visit(H)

Depth First Search

```
OS stack
    dfs_visit(H)
    dfs_visit(E)
    dfs_visit(C)
```

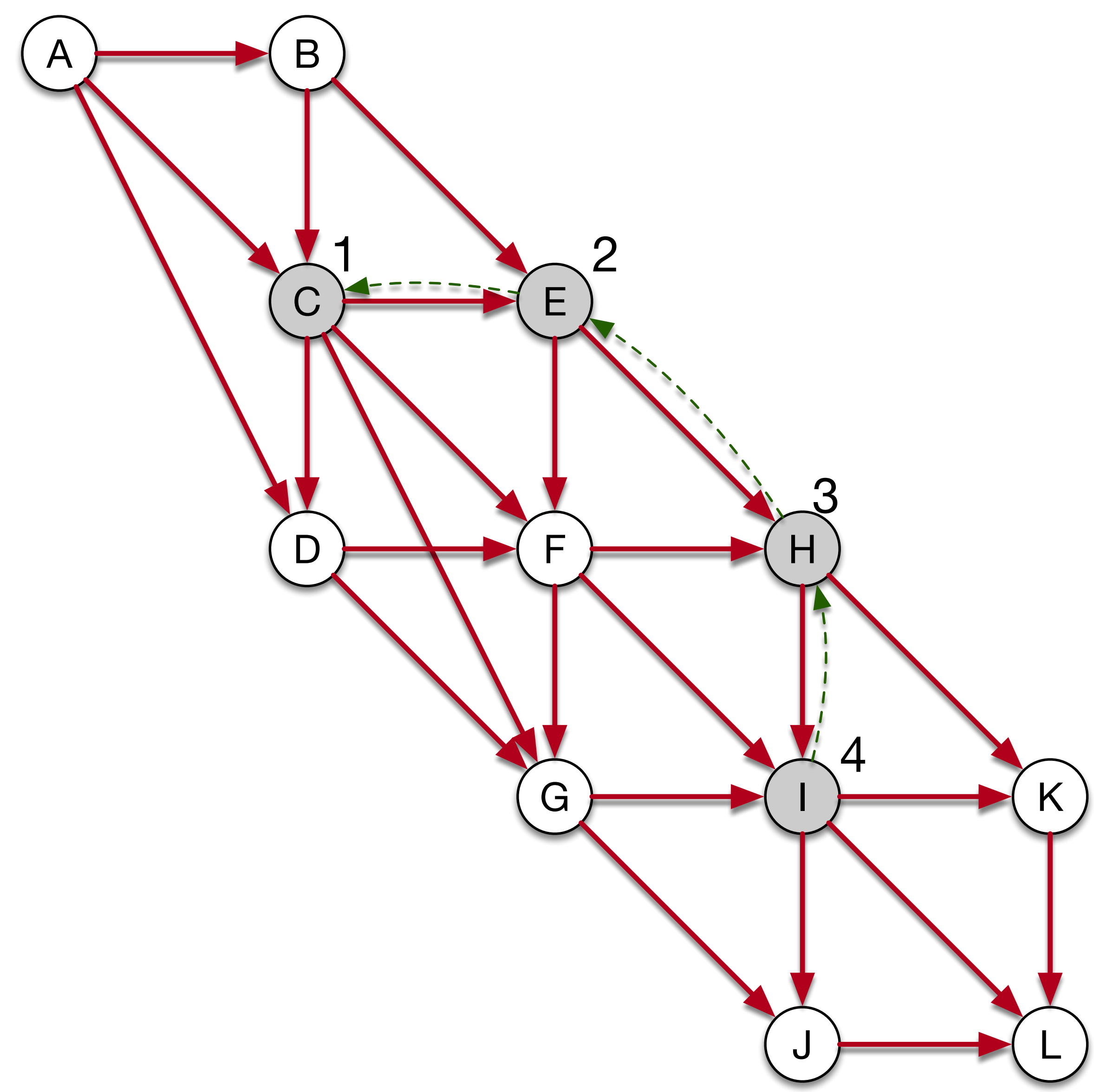
```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```




```
we pick v = H and call
dfs_visit(H)
this colors H gray
```

Depth First Search

```
OS stack
    dfs_visit(I)
    dfs_visit(H)
    dfs_visit(E)
    dfs_visit(C)
```

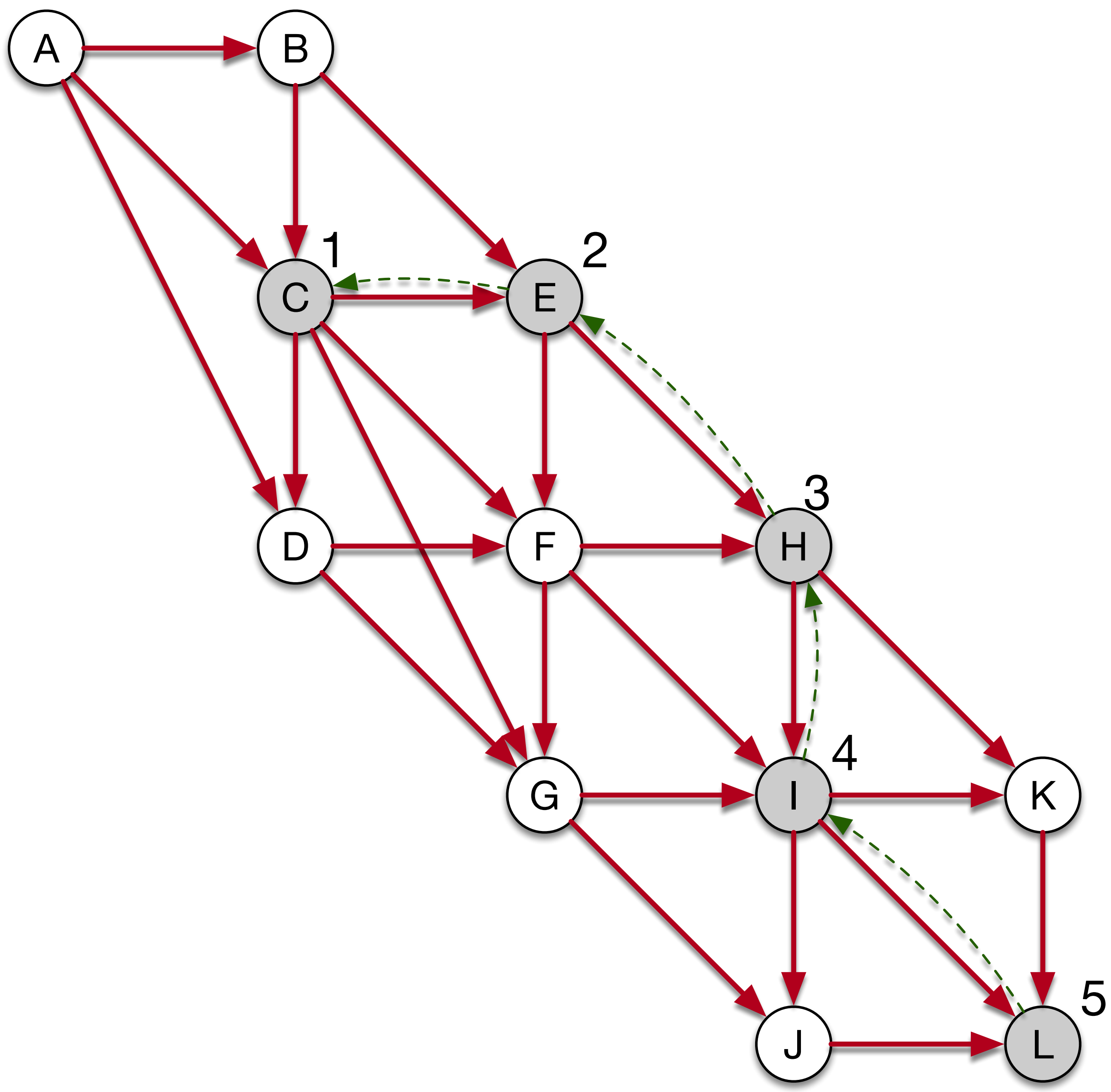



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

```
we pick v = I and call
dfs_visit(I)
this colors I gray
```

Depth First Search

```
OS stack
    dfs_visit(L)
    dfs_visit(I)
    dfs_visit(H)
    dfs_visit(E)
    dfs_visit(C)
```

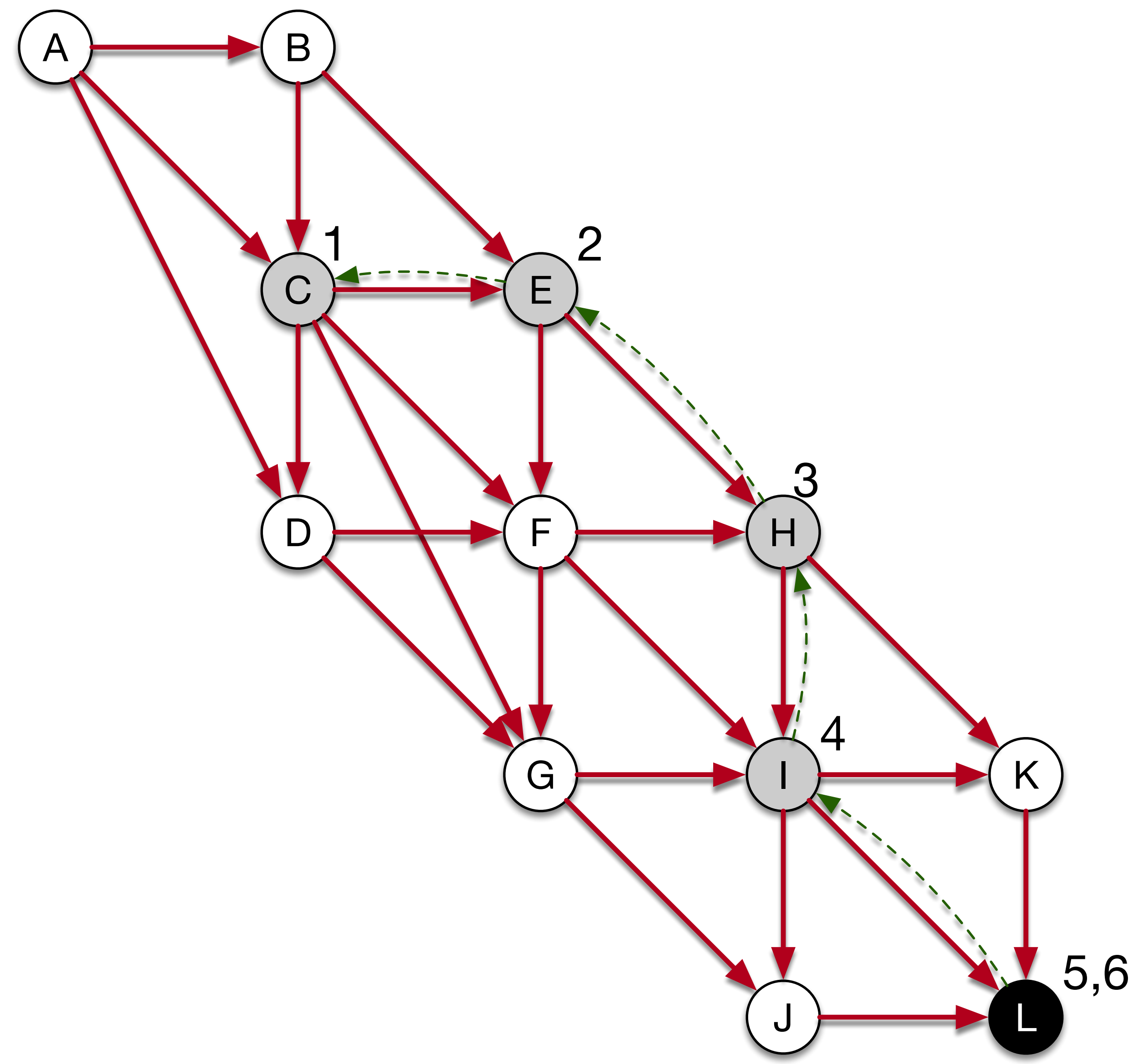



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

```
we pick v = L and call
dfs_visit(L)
this colors L gray
```

Depth First Search

```
OS stack
    dfs_visit(L)
    dfs_visit(I)
    dfs_visit(H)
    dfs_visit(E)
    dfs_visit(C)
```

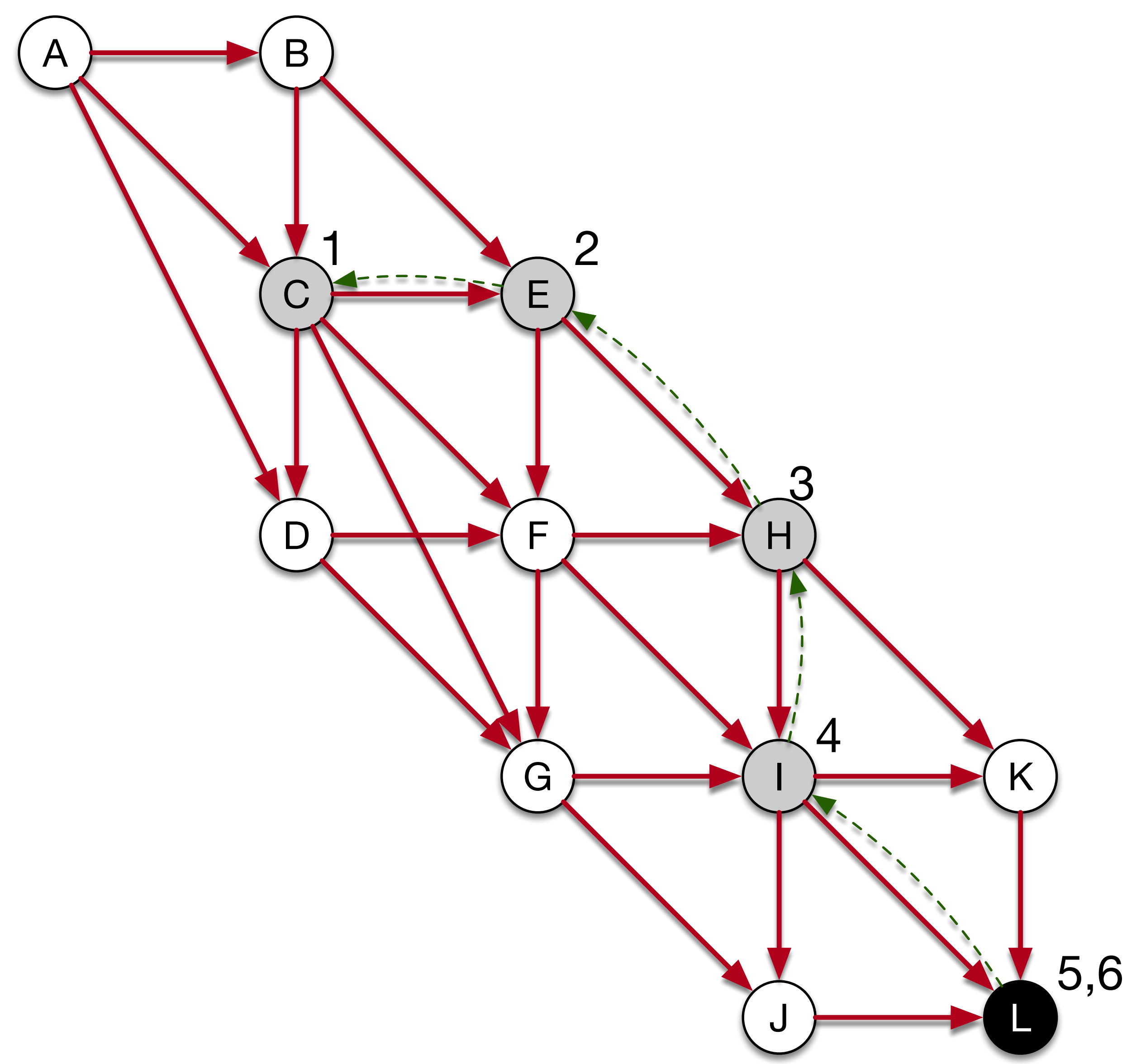



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

L has no vertices in the adjacency list
Therefore, we finally go to the last line

Depth First Search

OS stack
    dfs_visit(I)
    dfs_visit(H)
    dfs_visit(E)
    dfs_visit(C)

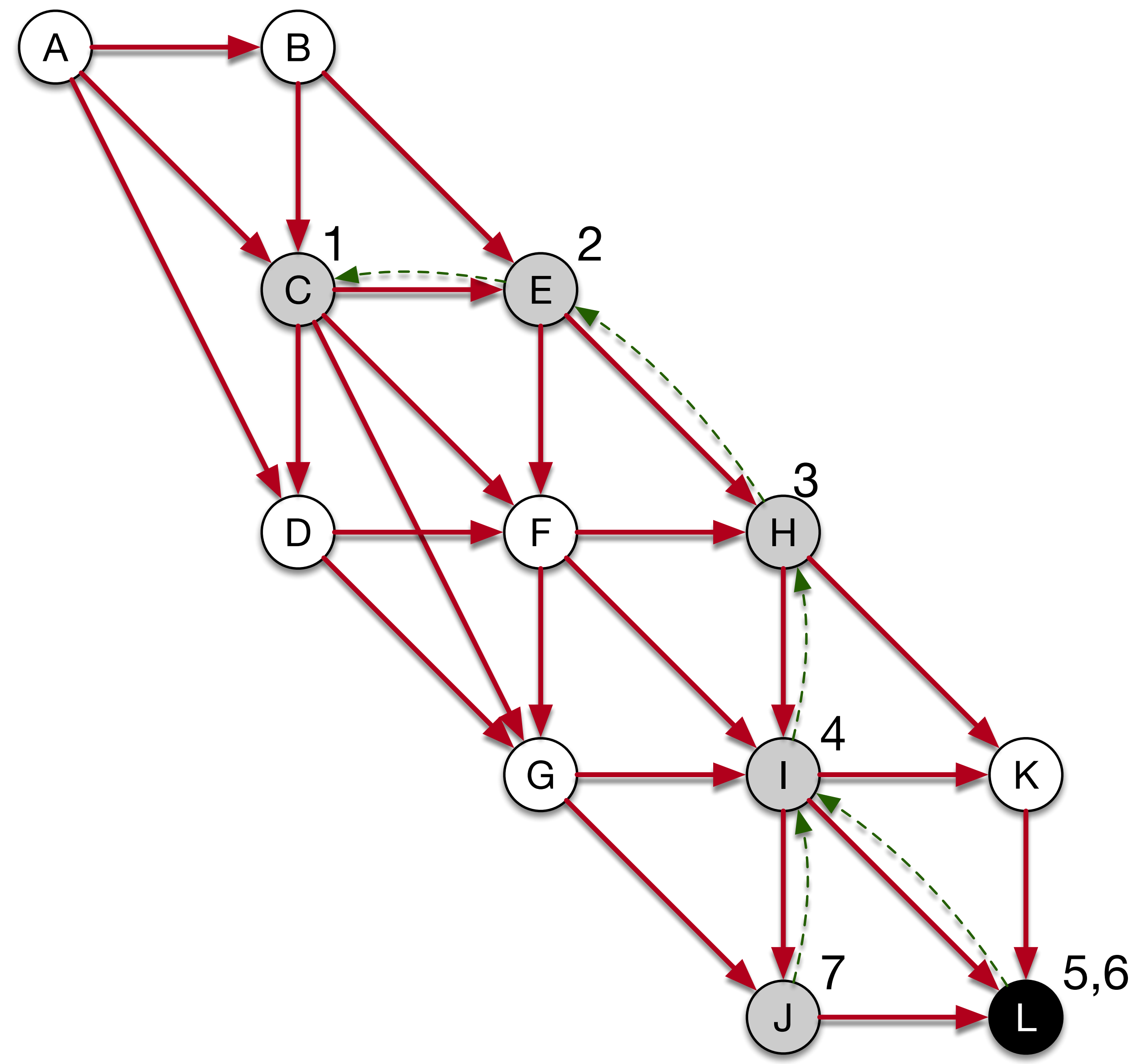



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

We finish dfs_visit(L) and are back at the execution of dfs_visit(I)

Depth First Search

```
OS stack
    dfs_visit(J)
    dfs_visit(I)
    dfs_visit(H)
    dfs_visit(E)
    dfs_visit(C)
```

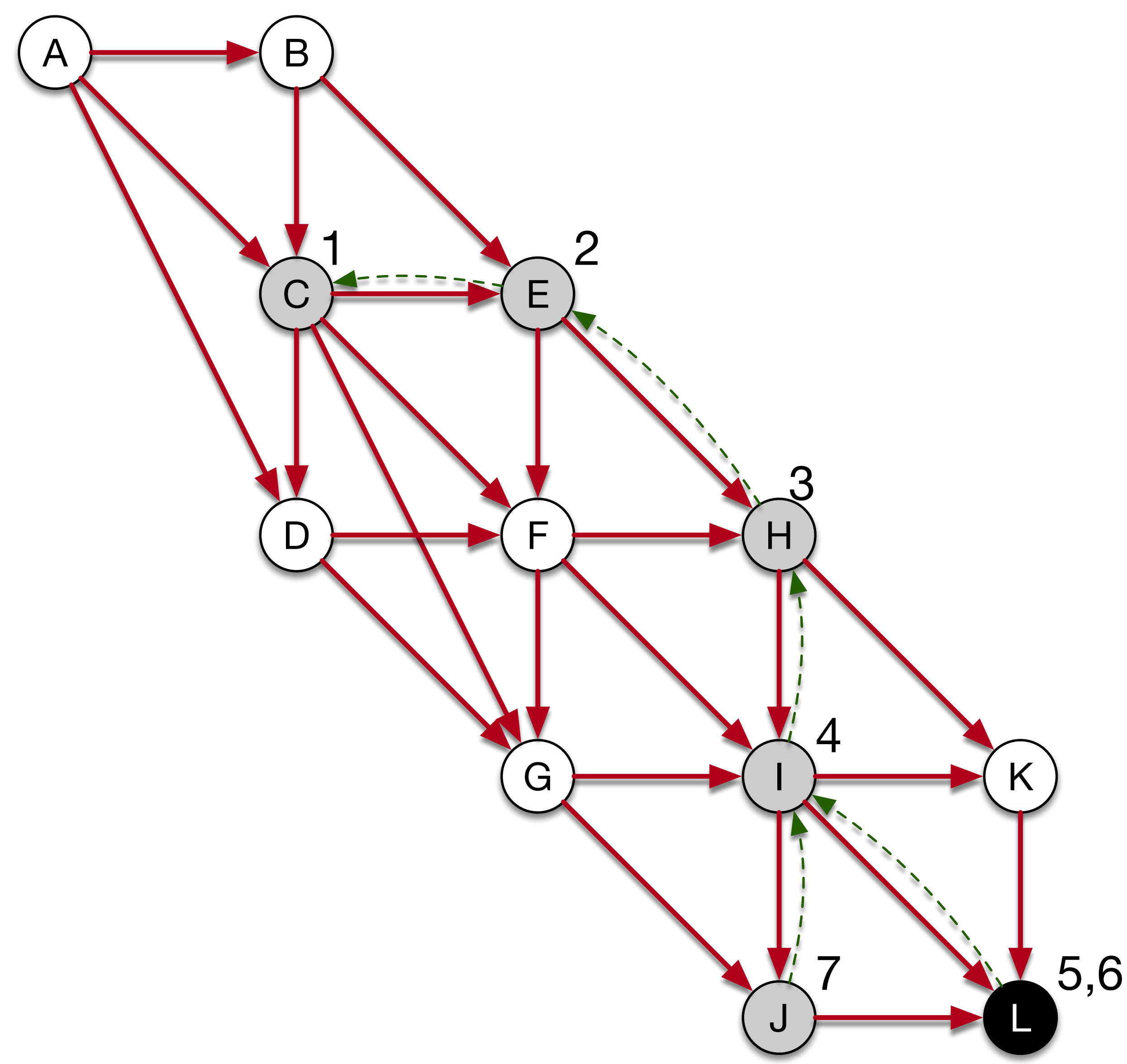



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

We pick a white vertex reachable from I: J

Depth First Search

```
OS stack
    dfs_visit(J)
    dfs_visit(I)
    dfs_visit(H)
    dfs_visit(E)
    dfs_visit(C)
```

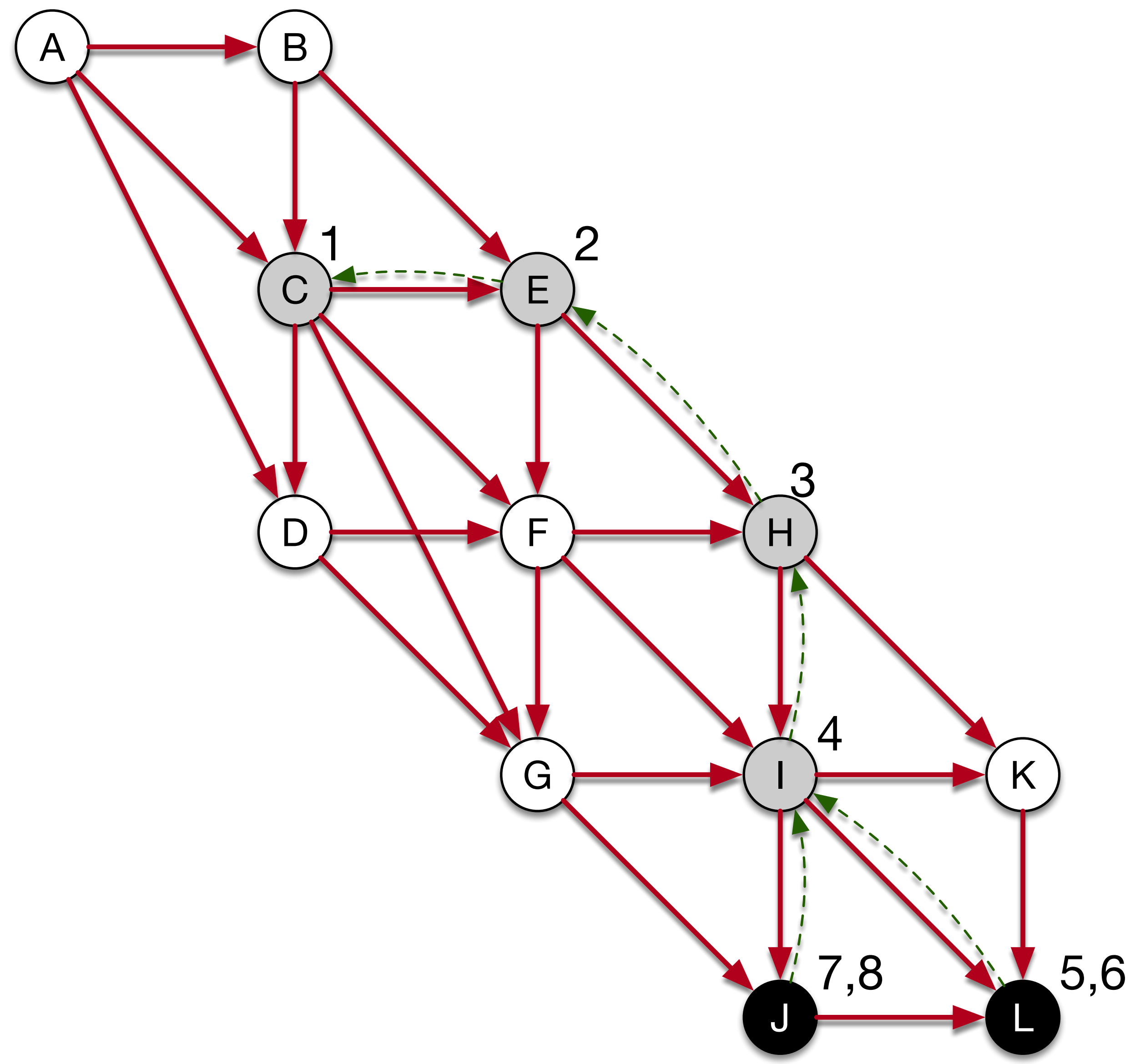



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

There are no white nodes in the adjacency list of J

Depth First Search

```
OS stack
    dfs_visit(I)
    dfs_visit(H)
    dfs_visit(E)
    dfs_visit(C)
```

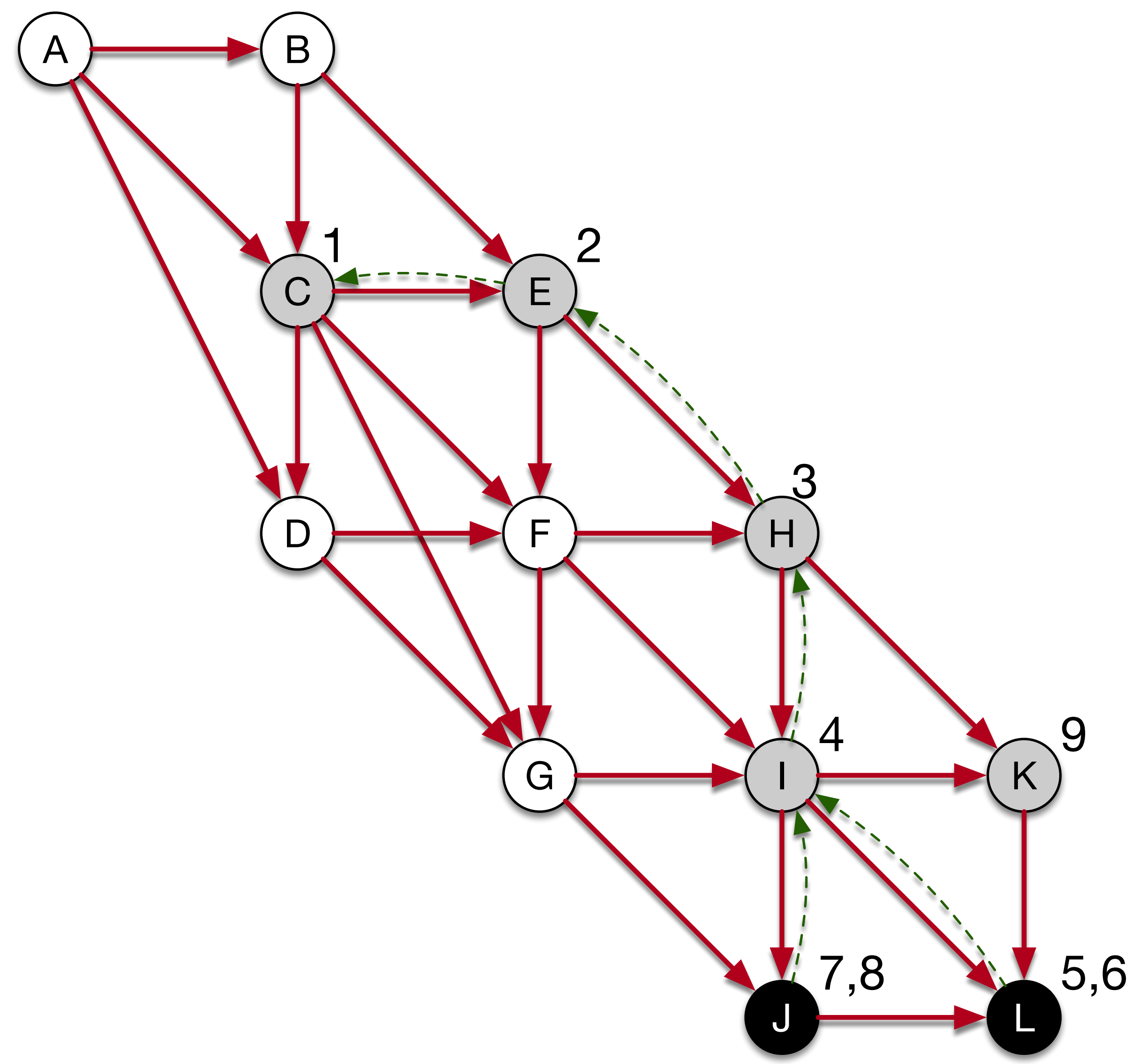



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

We close the call on J and are back to dfs_visit(I)

Depth First Search

```
OS stack
    dfs_visit(K)
    dfs_visit(I)
    dfs_visit(H)
    dfs_visit(E)
    dfs_visit(C)
```



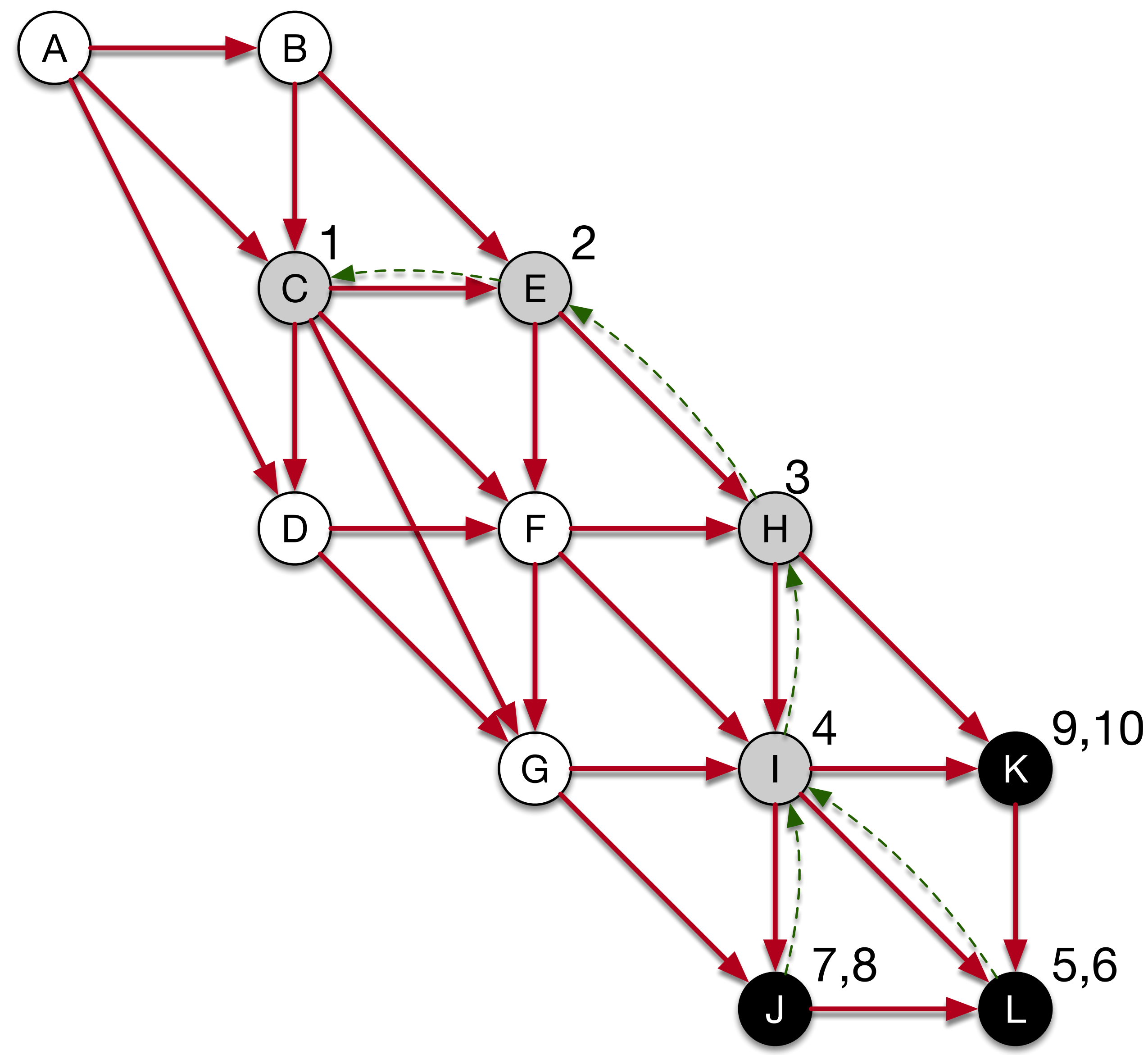

```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

dfs_visit(I) now goes to K

Depth First Search

```
OS stack
    dfs_visit(I)
    dfs_visit(H)
    dfs_visit(E)
    dfs_visit(C)
```

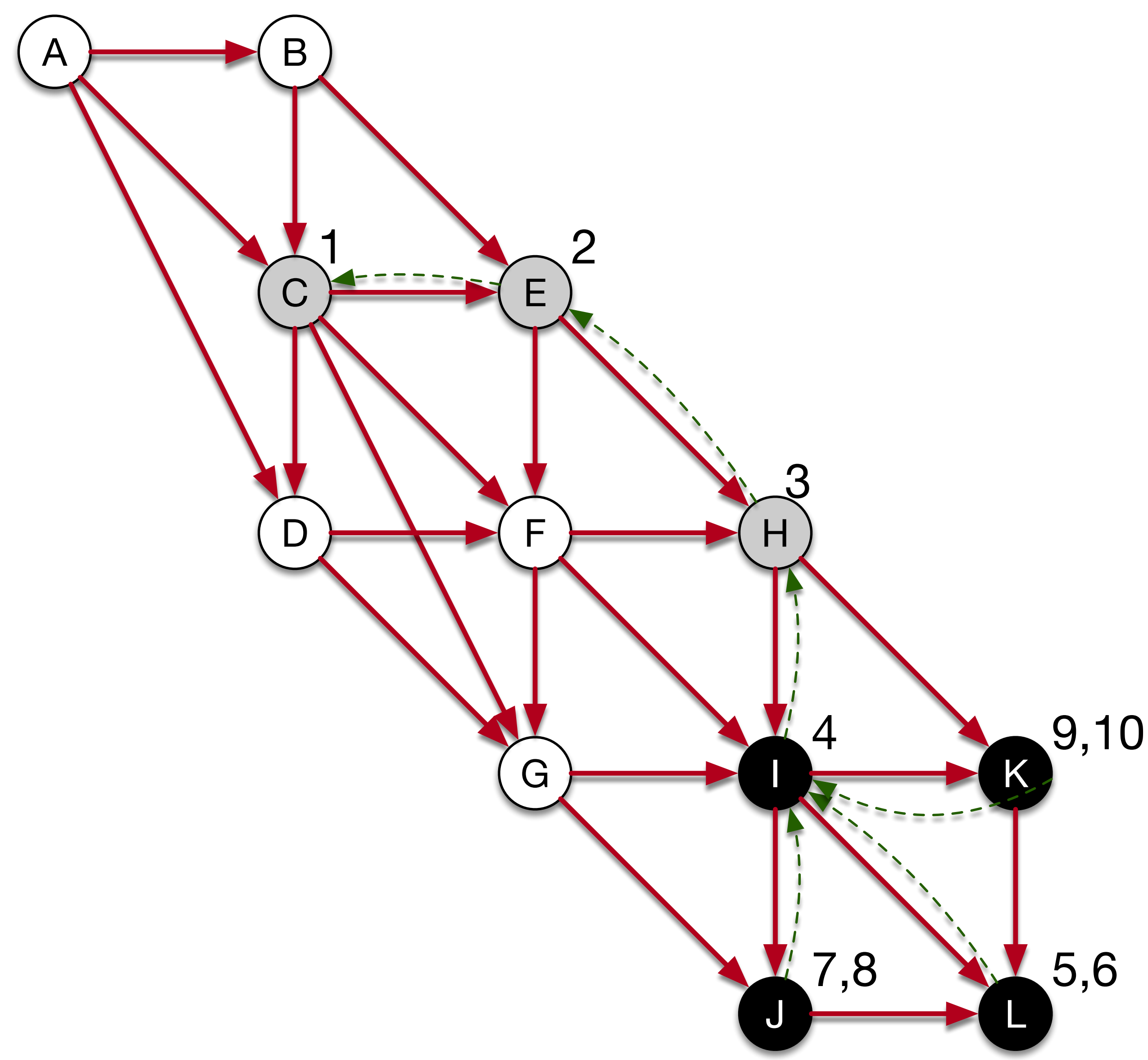
```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```



`dfs_visit(K) finishes`

Depth First Search

```
OS stack
   dfs_visit(H)
   dfs_visit(E)
   dfs_visit(C)
```

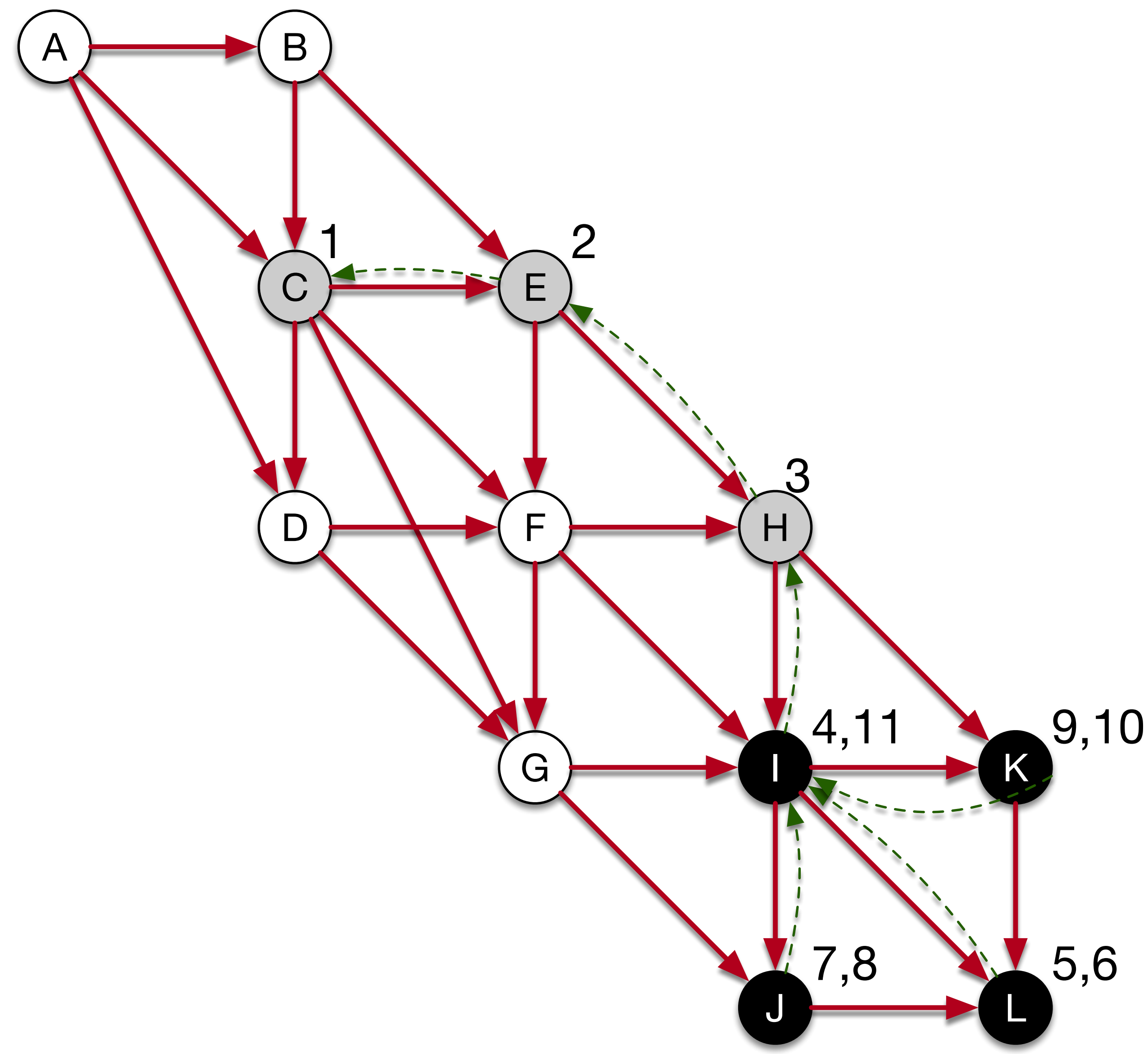



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

```
dfs_visit(I) runs again
but finds no white vertices,
so it finishes
```

Depth First Search

OS stack
    dfs_visit(H)
    dfs_visit(E)
    dfs_visit(C)

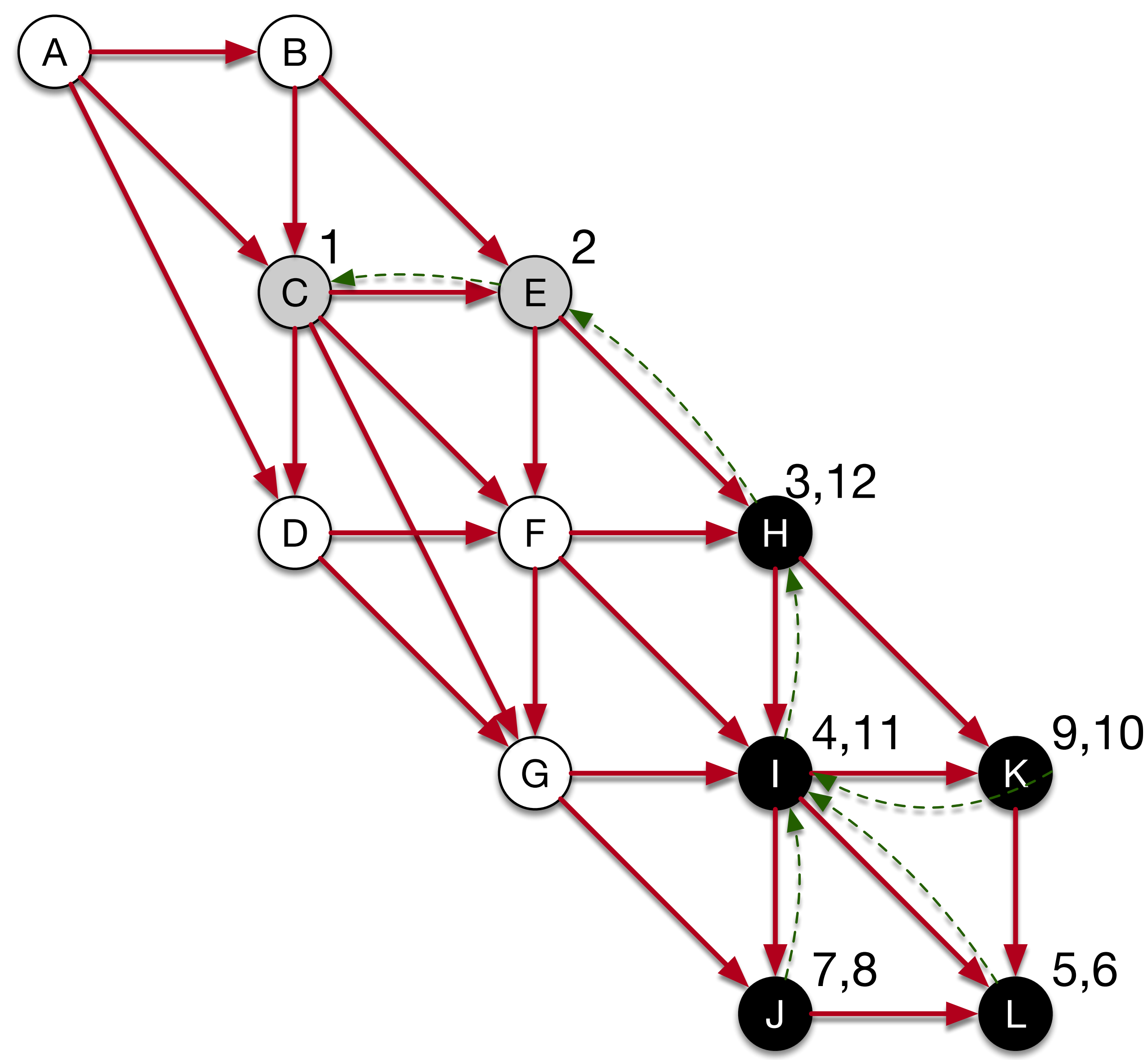



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

dfs_visit(I) runs again but finds no white vertices, so it finishes

Depth First Search

```
OS stack
   dfs_visit(E)
   dfs_visit(C)
```

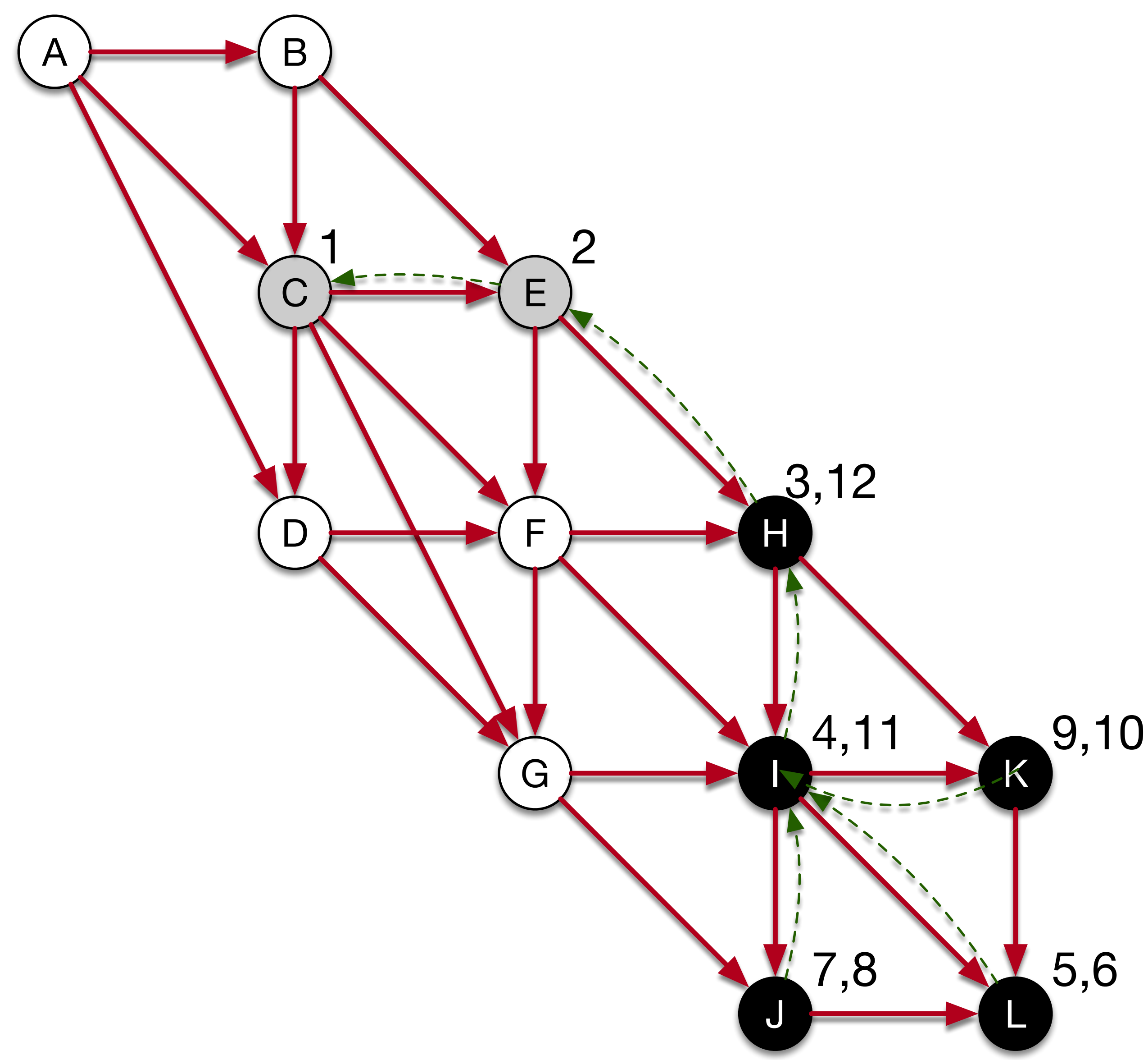



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

```
dfs_visit(H) runs again
but finds no white vertices,
so it finishes
```

Depth First Search

```
OS stack
   dfs_visit(E)
   dfs_visit(C)
```

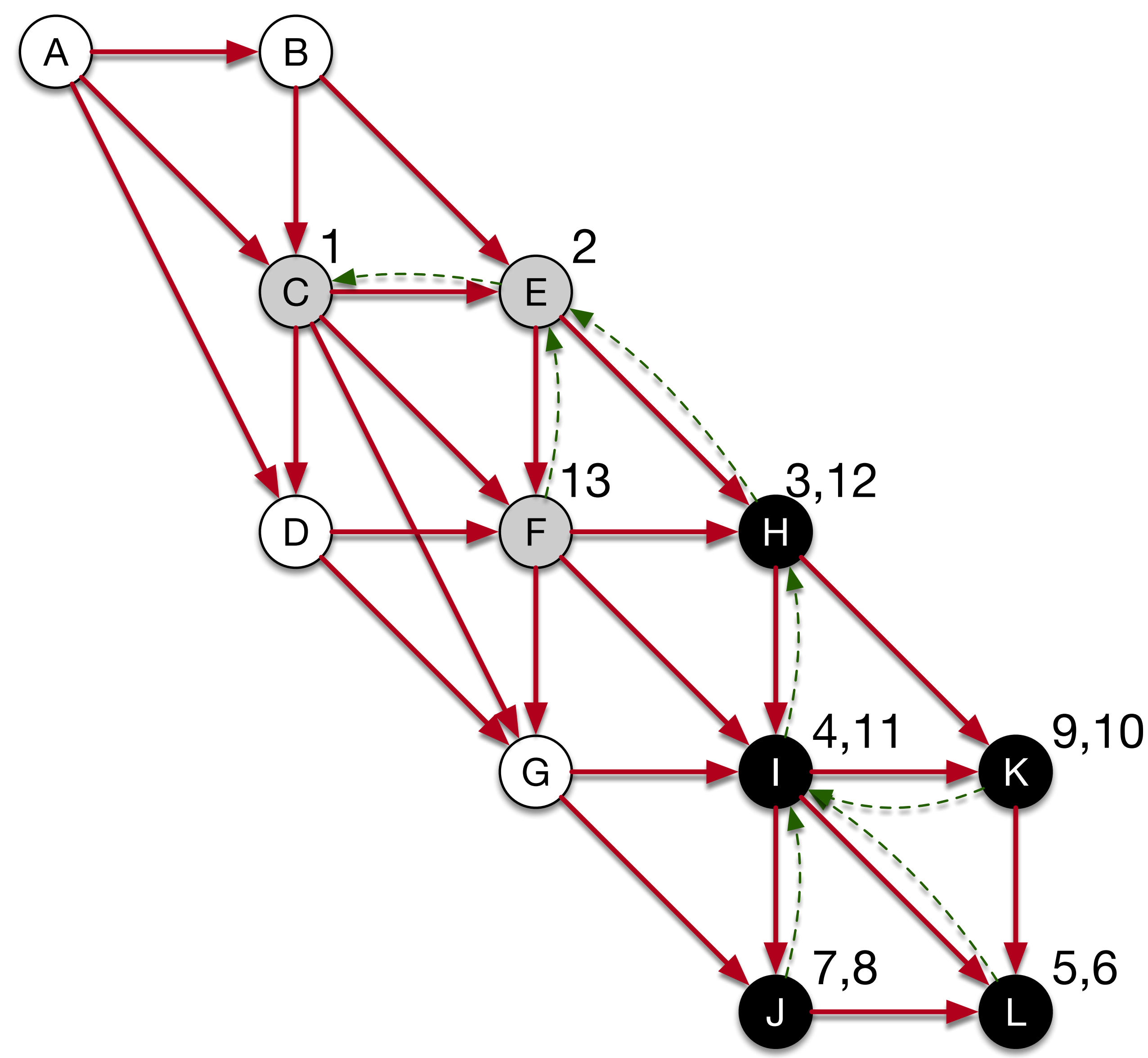



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

dfs_visit(E) runs again

Depth First Search

```
OS stack
    dfs_visit(F)
    dfs_visit(E)
    dfs_visit(C)
```

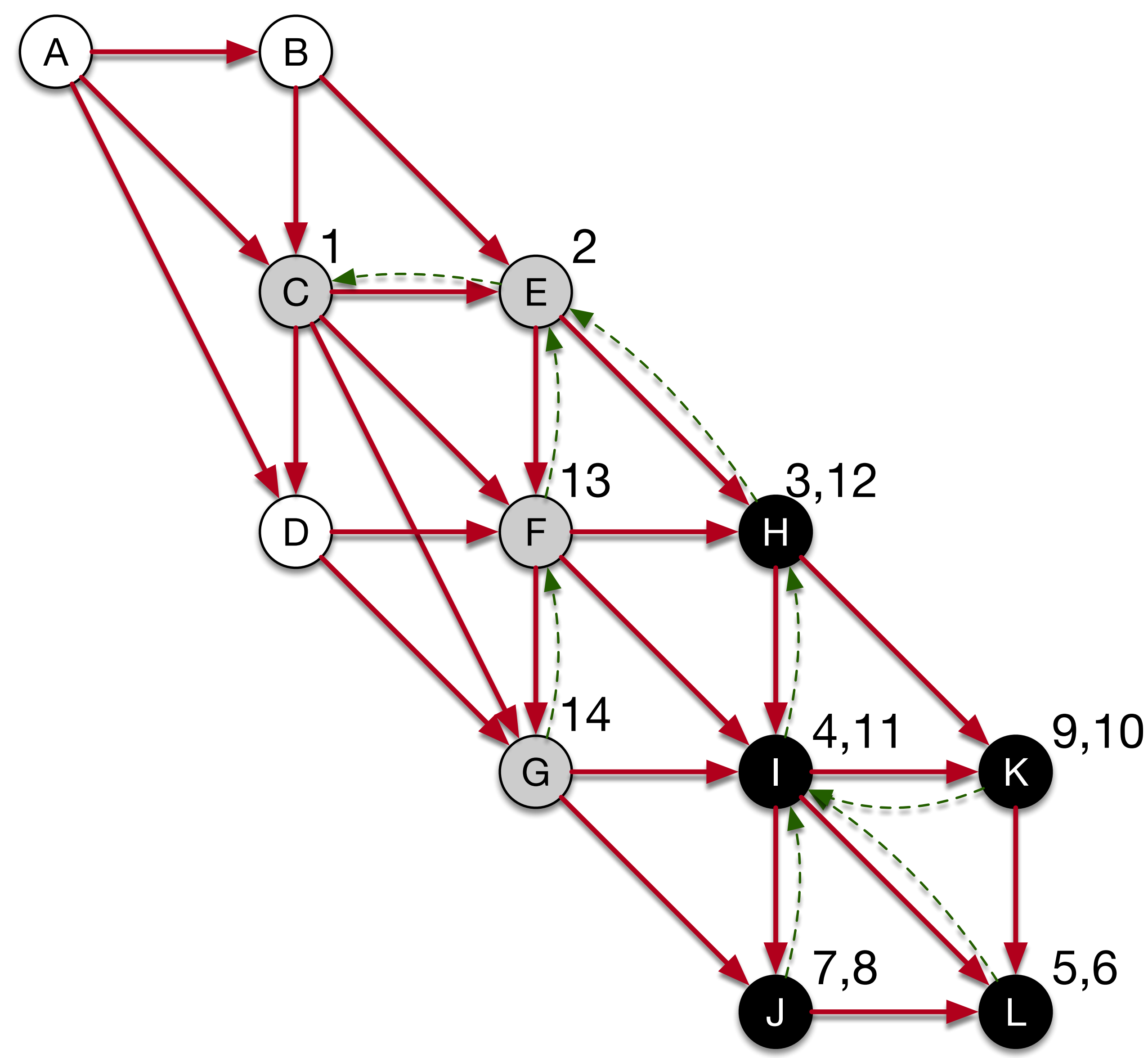



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

Finds F

Depth First Search

```
OS stack
    dfs_visit(G)
    dfs_visit(F)
    dfs_visit(E)
    dfs_visit(C)
```

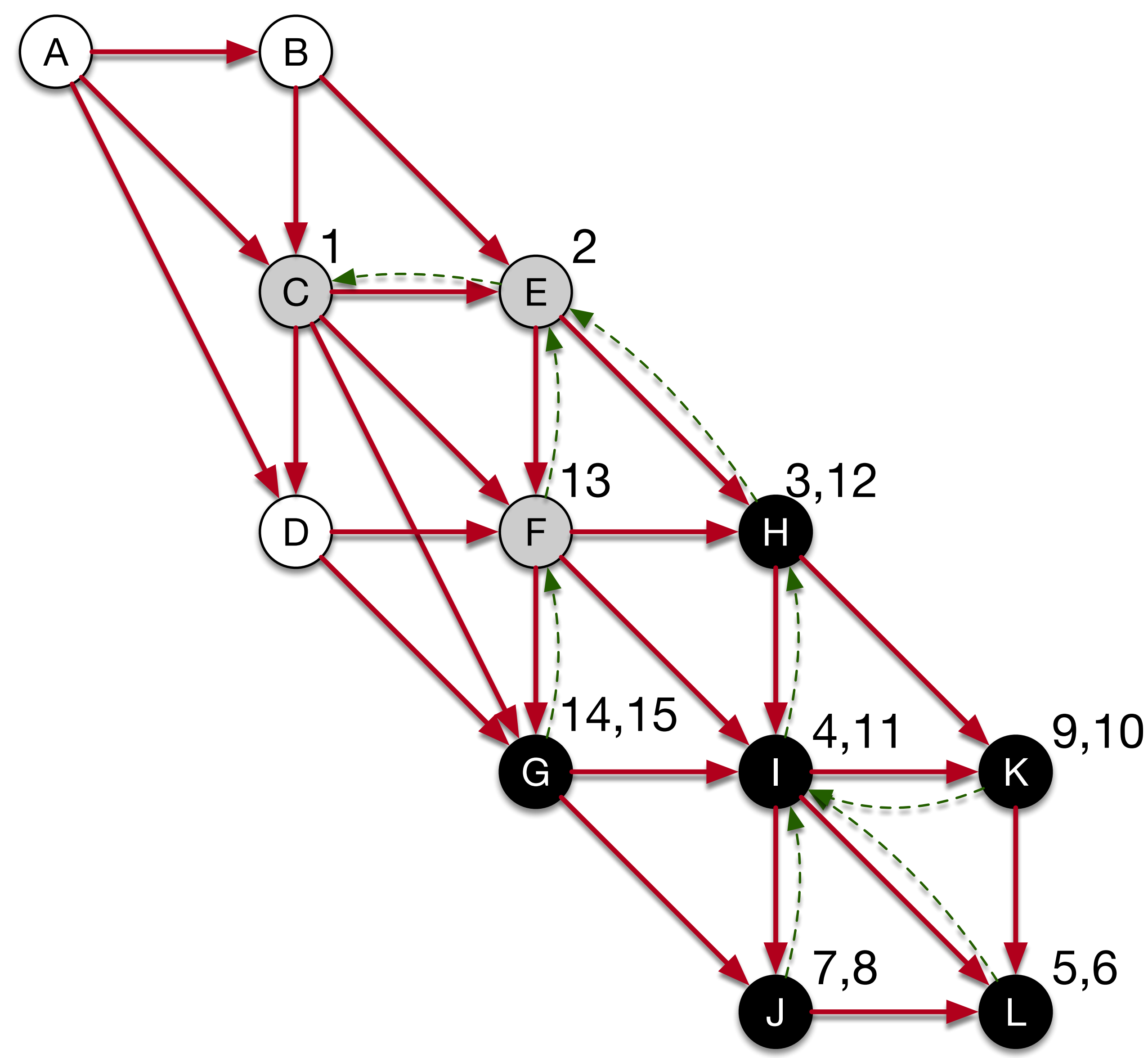



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

Finds G

Depth First Search

```
OS stack
    dfs_visit(F)
    dfs_visit(E)
    dfs_visit(C)
```

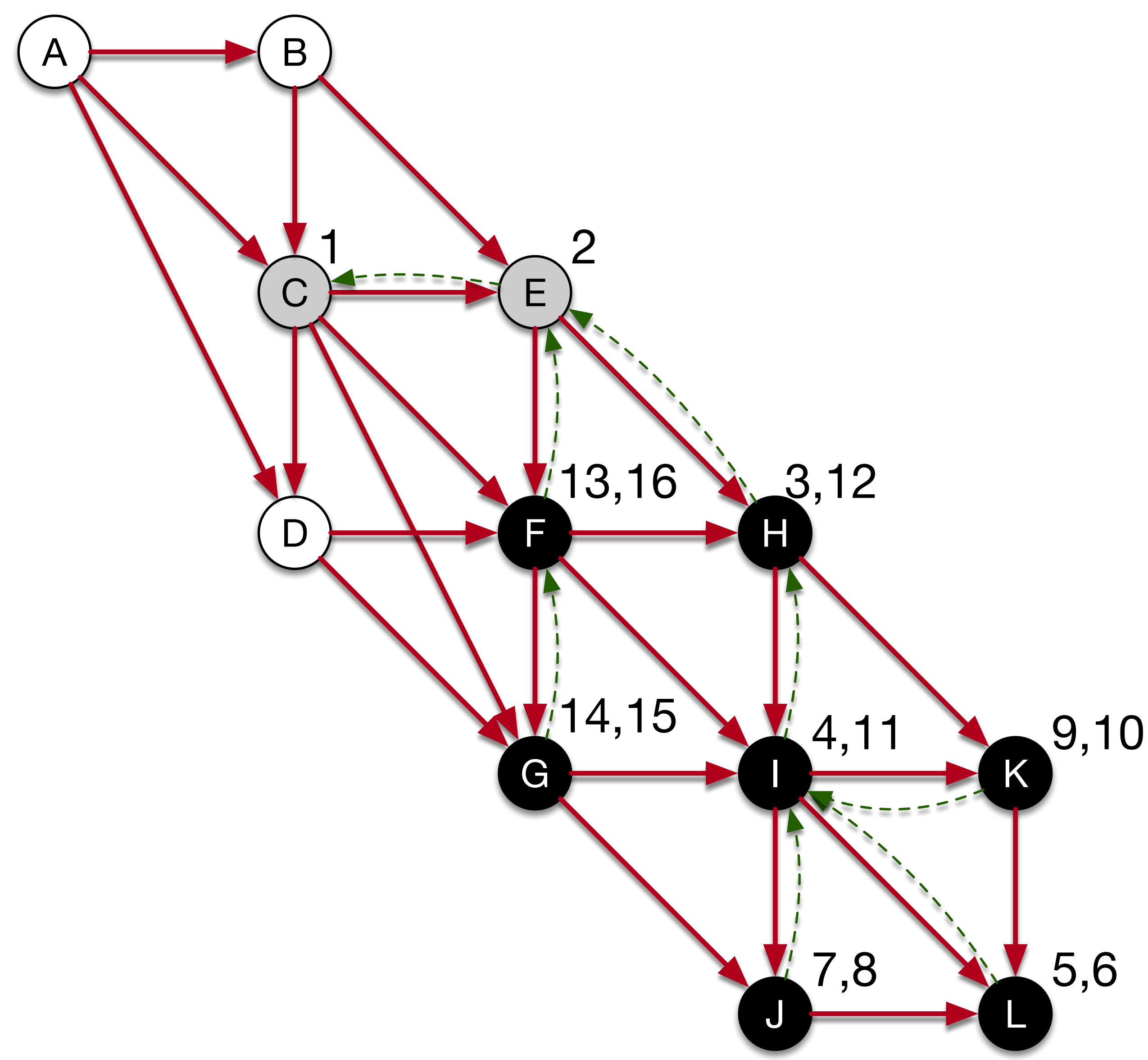



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

Nothing left in G

Depth First Search

```
OS stack
   dfs_visit(E)
   dfs_visit(C)
```

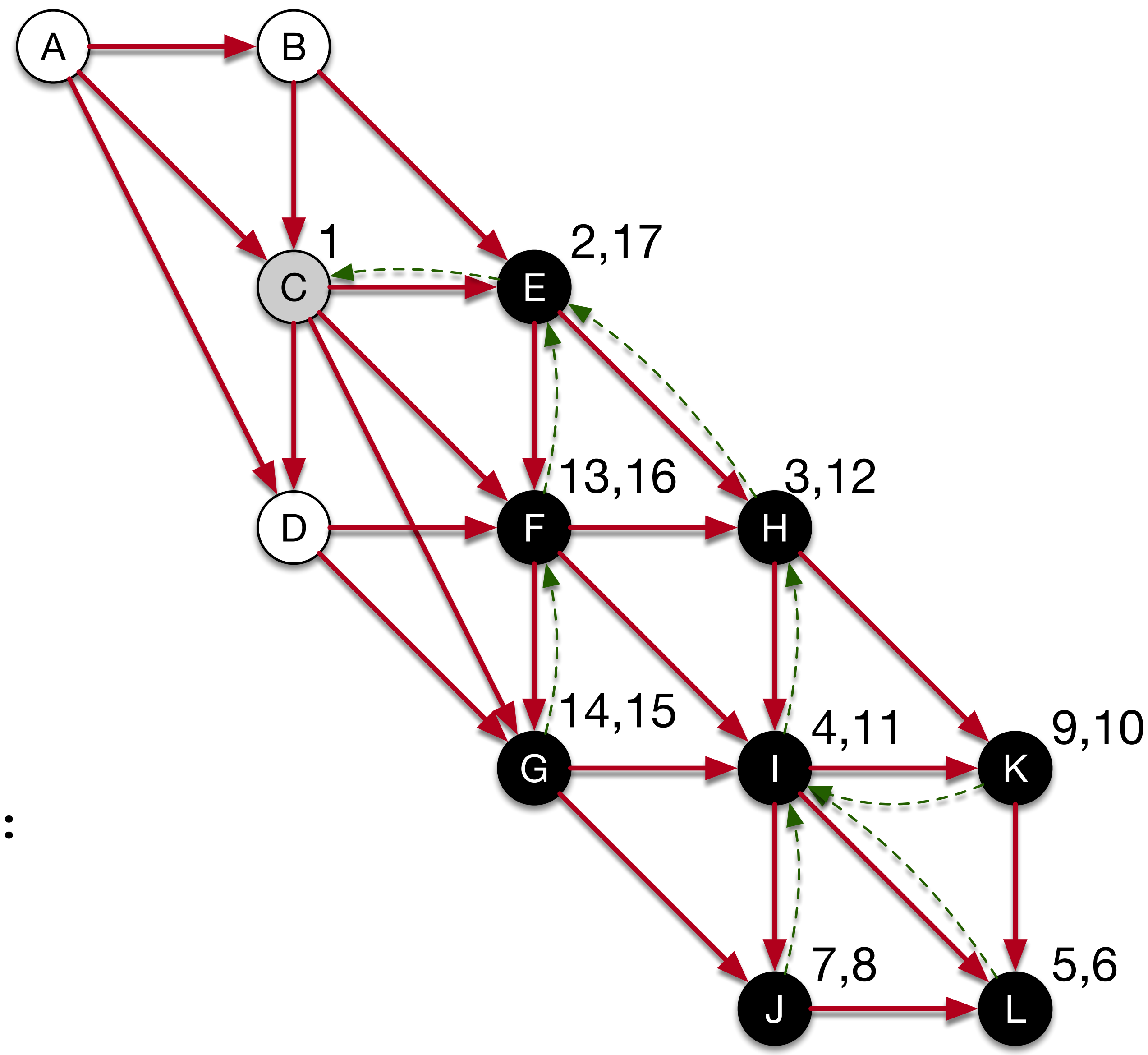



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

```
Nothing left in F
```

Depth First Search

```
OS stack
   dfs_visit(C)
```

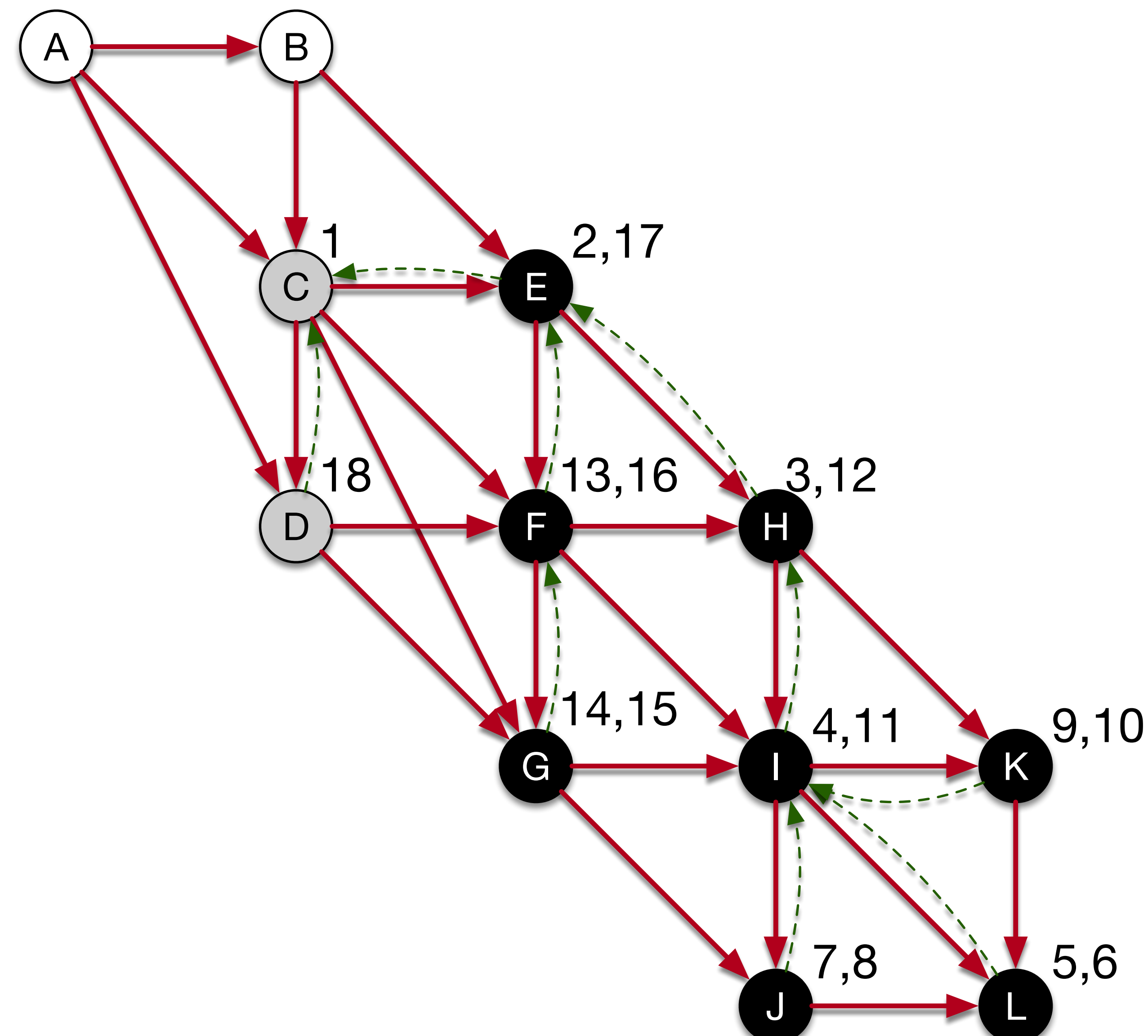



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

Finishing E

Depth First Search

```
OS stack
   dfs_visit(D)
   dfs_visit(C)
```

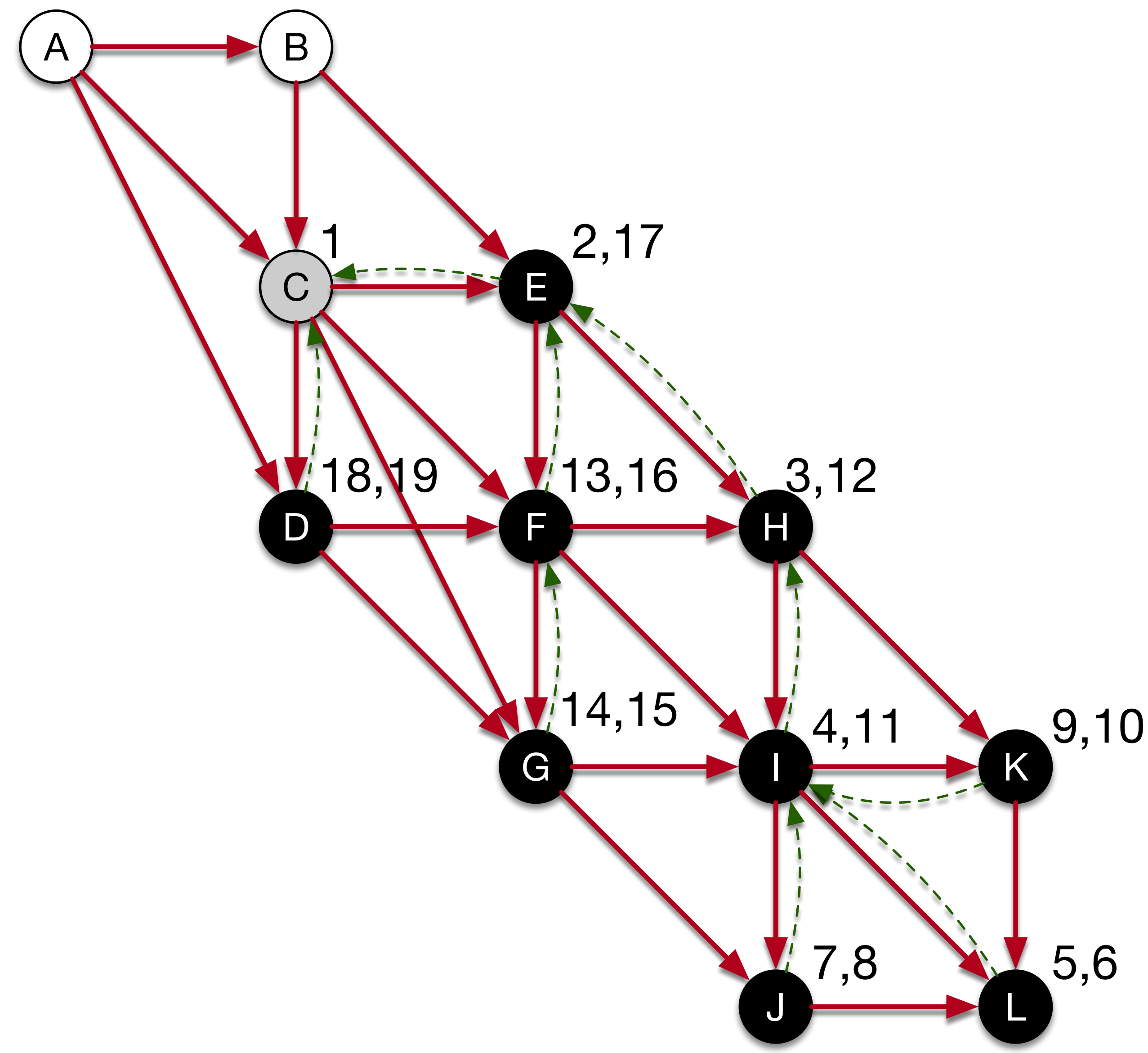



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

Last white vector in the adjacency list of C is D

Depth First Search

```
OS stack
   dfs_visit(D)
   dfs_visit(C)
```

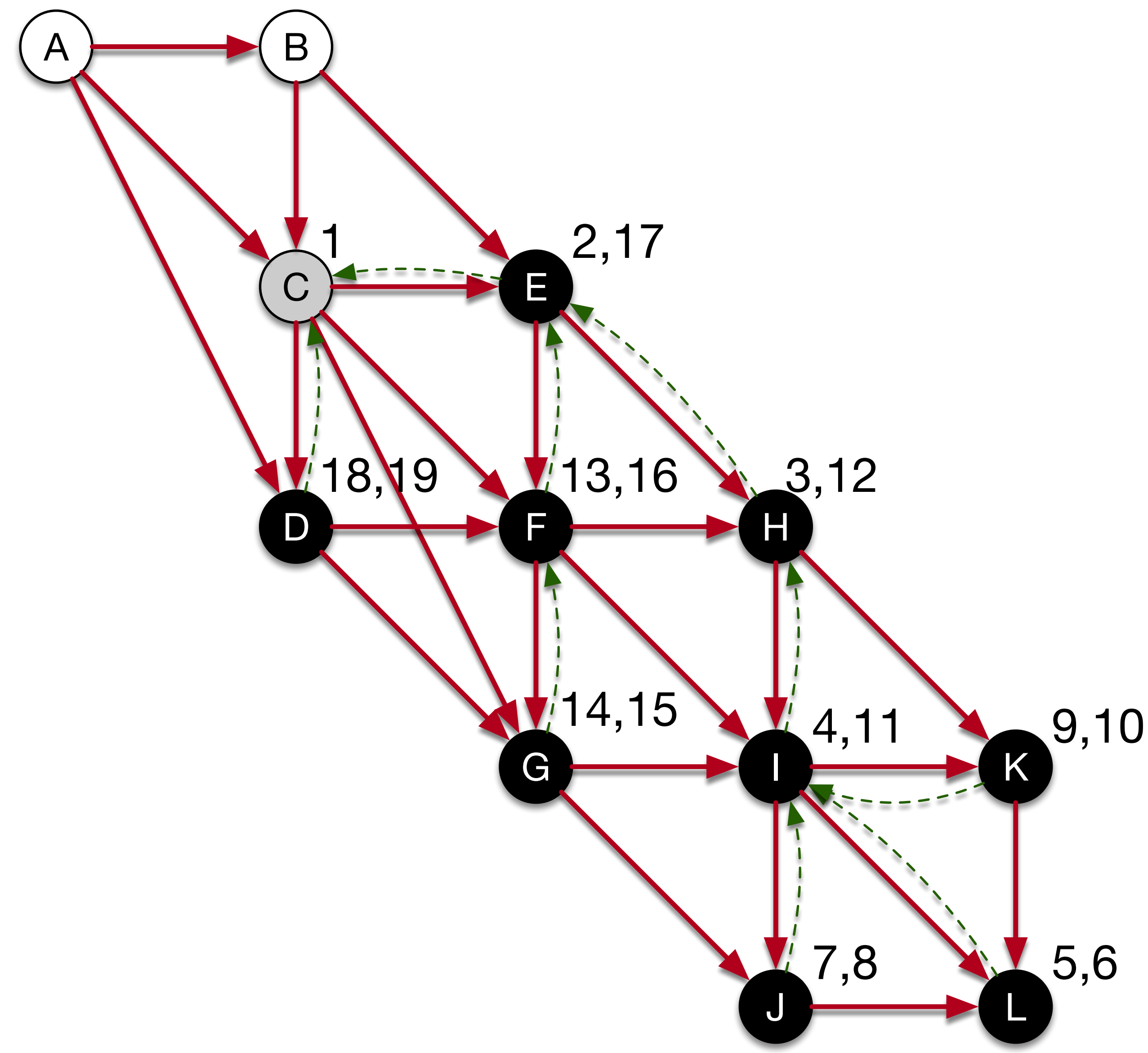



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

D has no white vertices in its adjacency list

Depth First Search

```
OS stack
    dfs_visit(C)
```

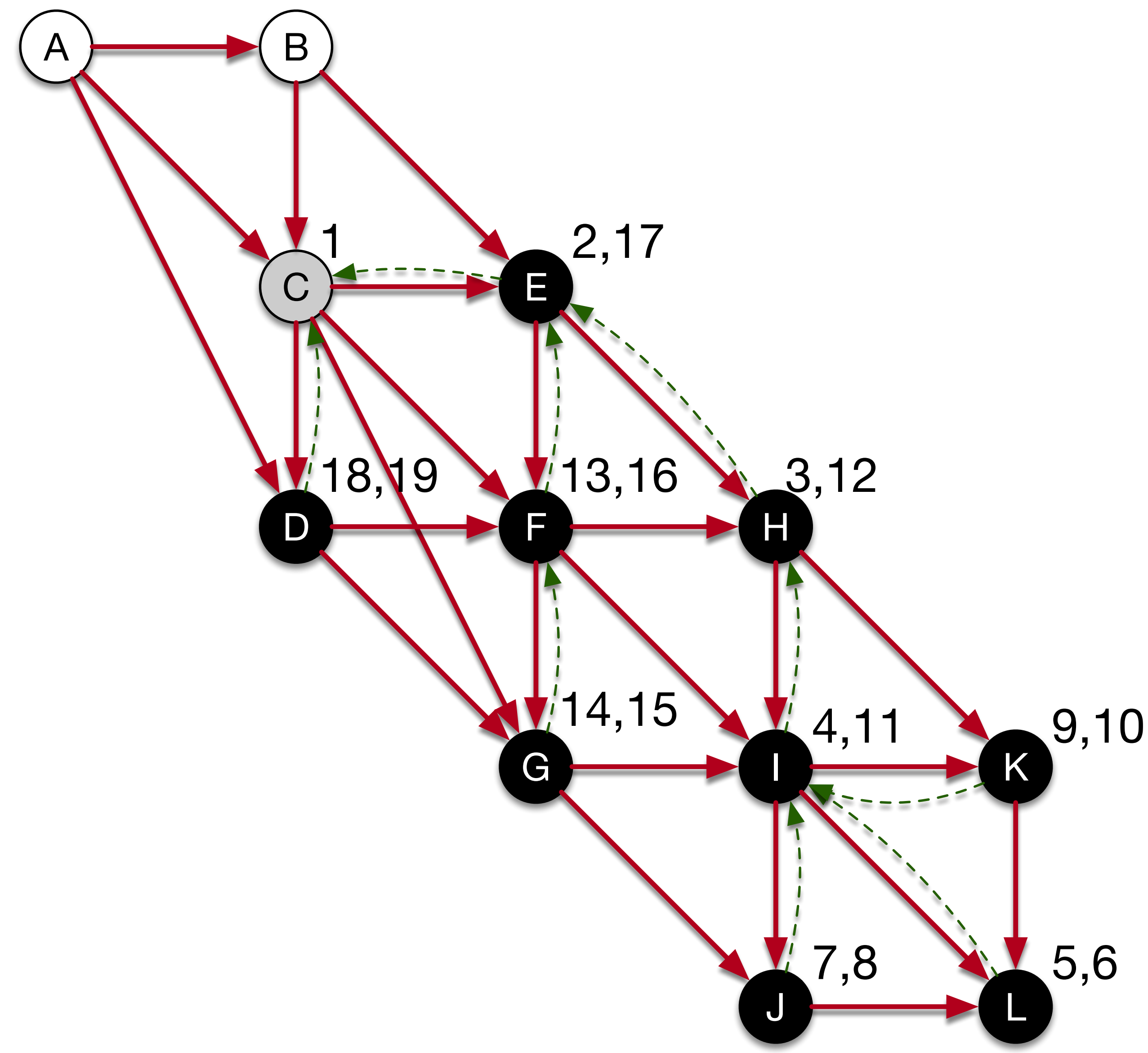



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

We are back in C

Depth First Search

```
OS stack
   dfs_visit(C)
```

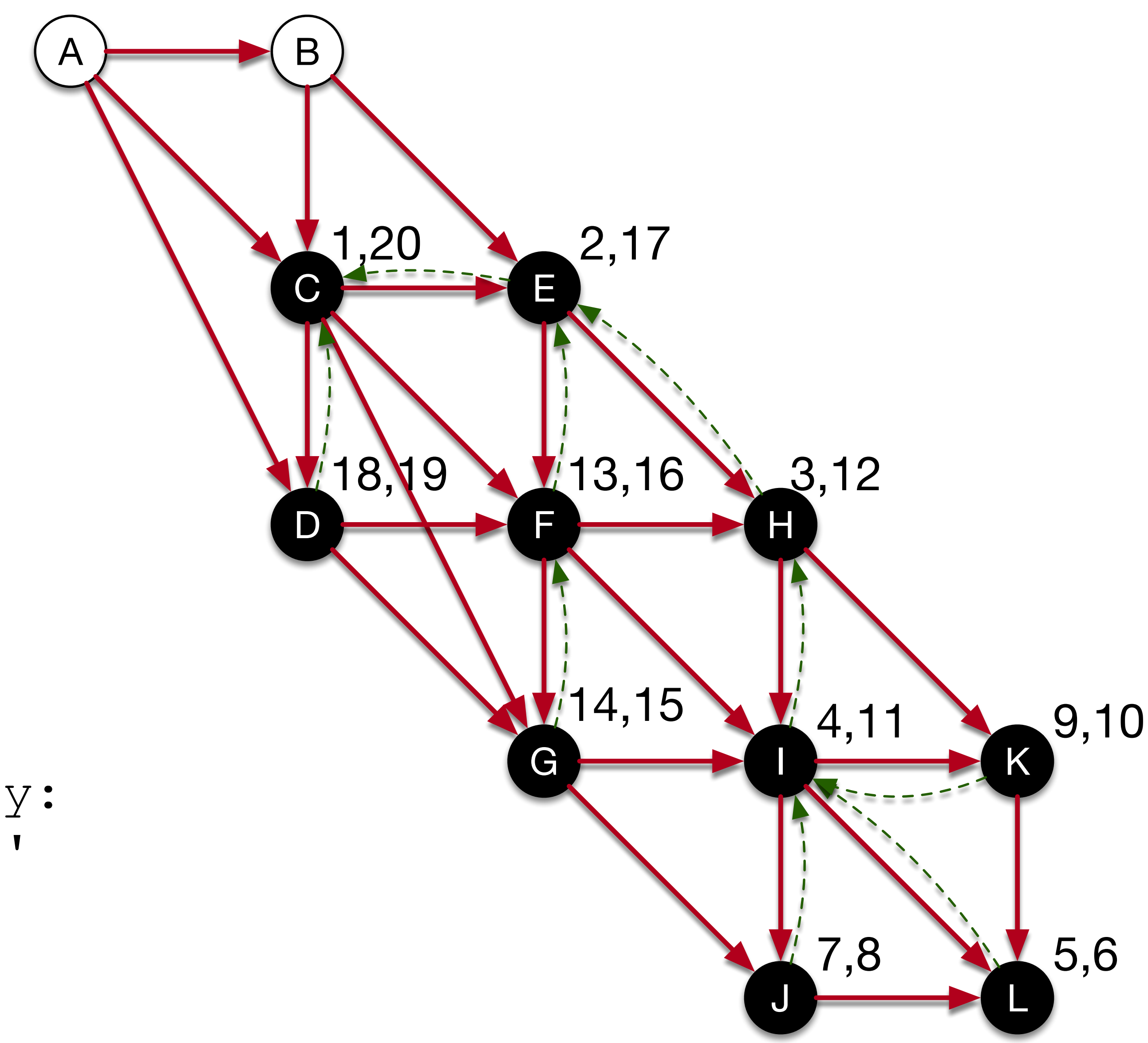



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

Now we close C

Depth First Search

OS stack




```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

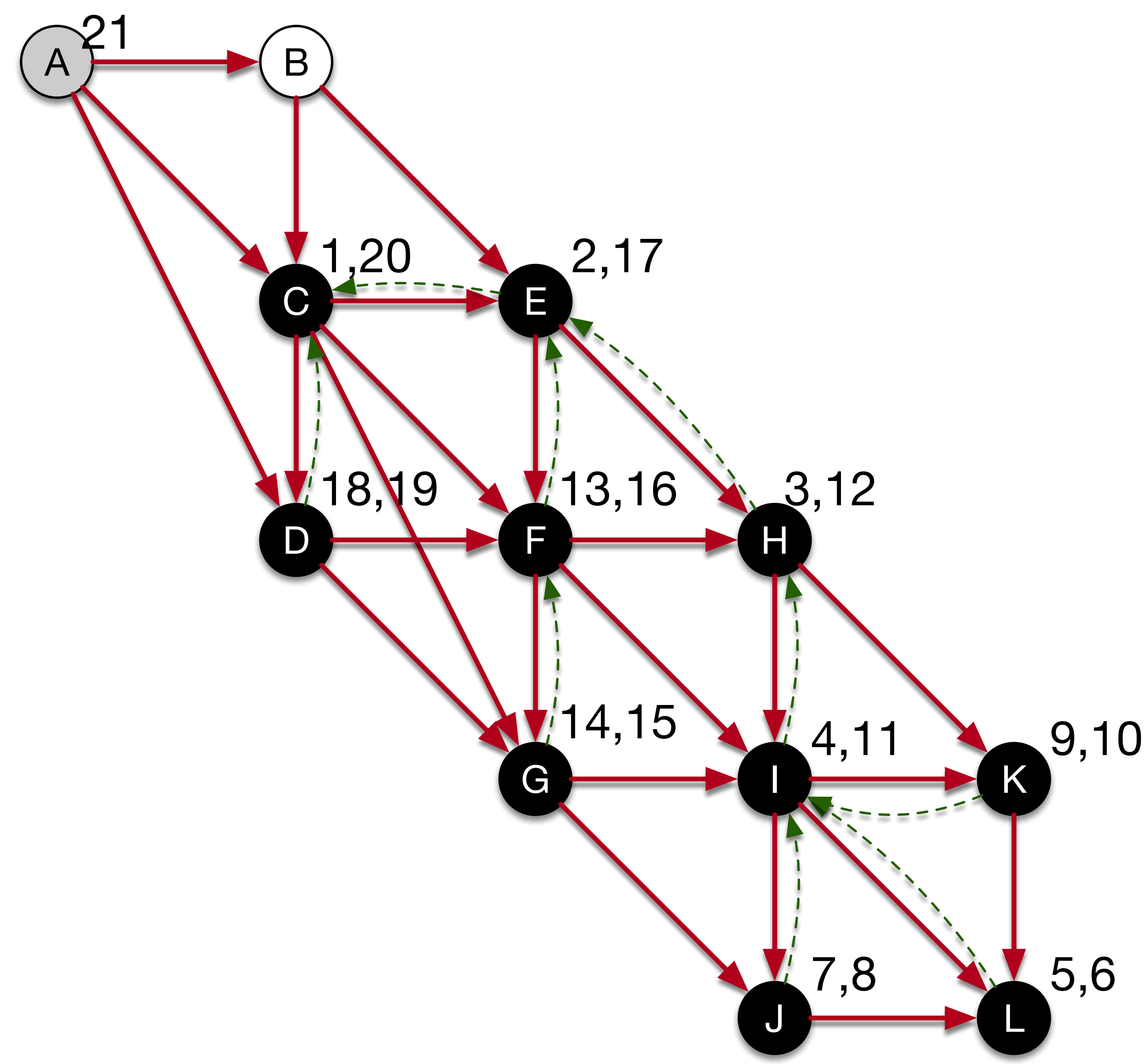
Now we close C

Depth First Search

- At this point, the original call to dfs_visit(C) is done
- However, since there are still white nodes left, we have to pick one of them and visit again.
- We pick A

Depth First Search

OS stack
    dfs_visit(A)

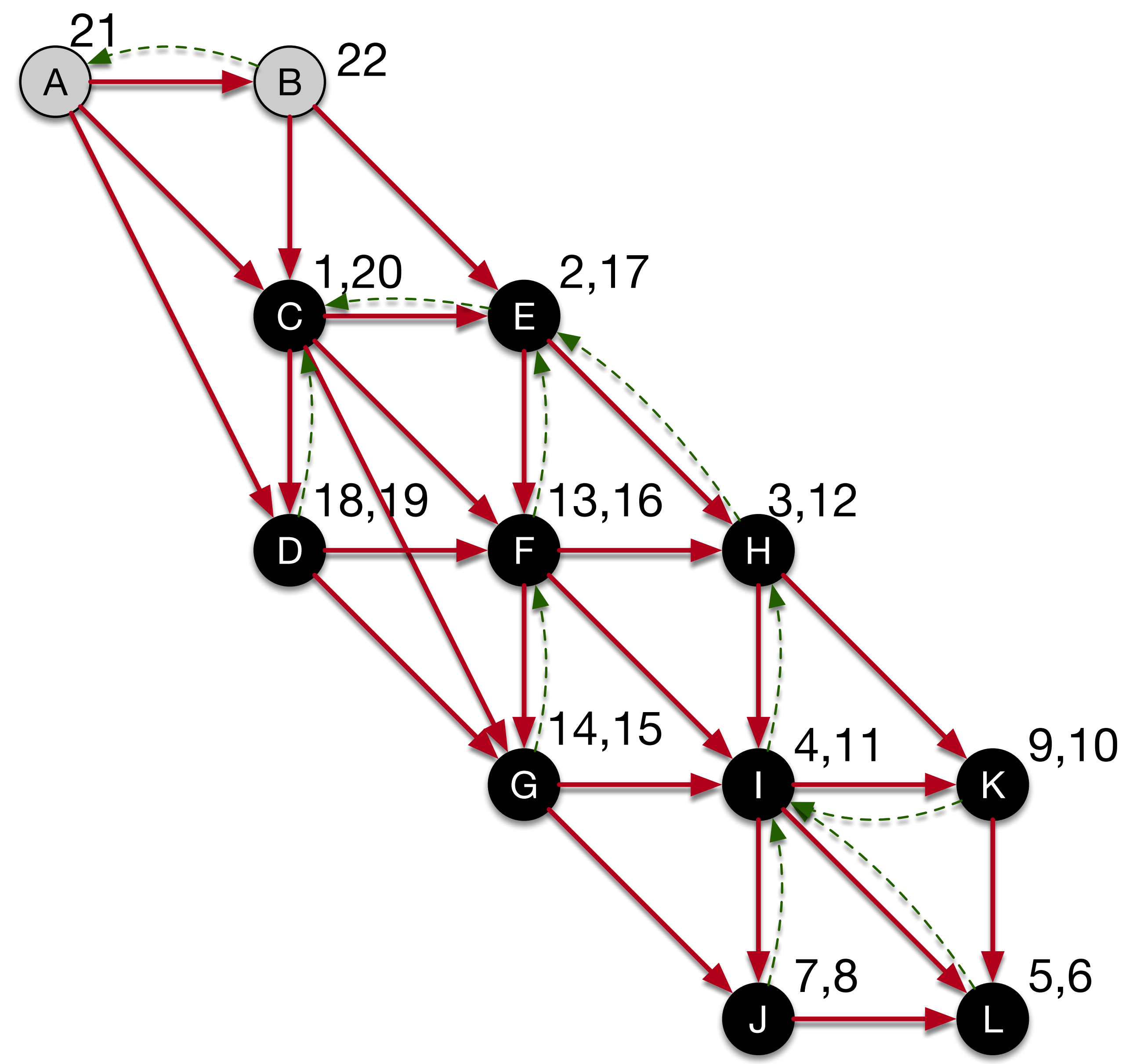



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

A is the only node in the stack

Depth First Search

```
OS stack
   dfs_visit(B)
   dfs_visit(A)
```

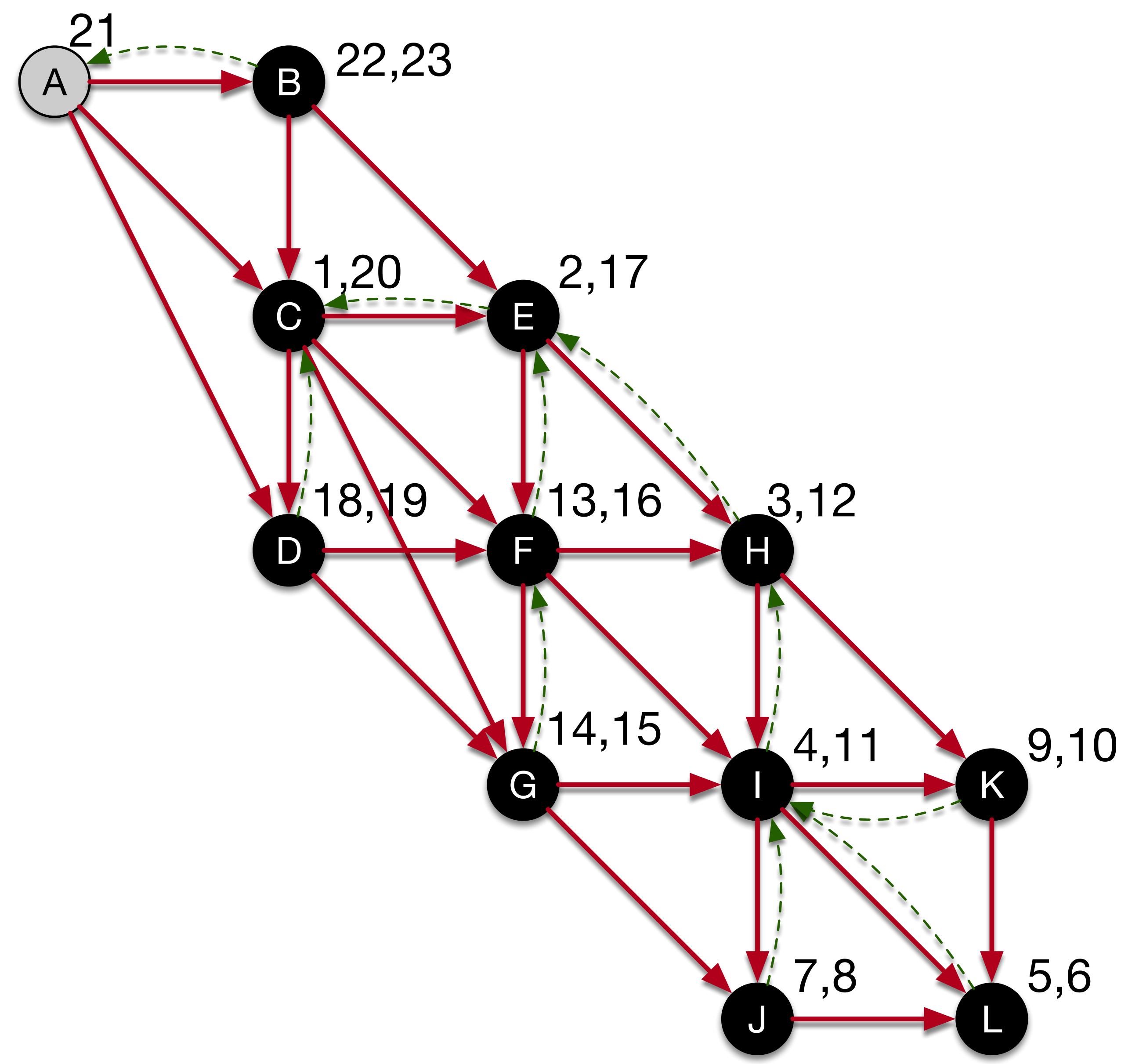



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

We discover B from A

Depth First Search

OS stack
    dfs_visit(A)

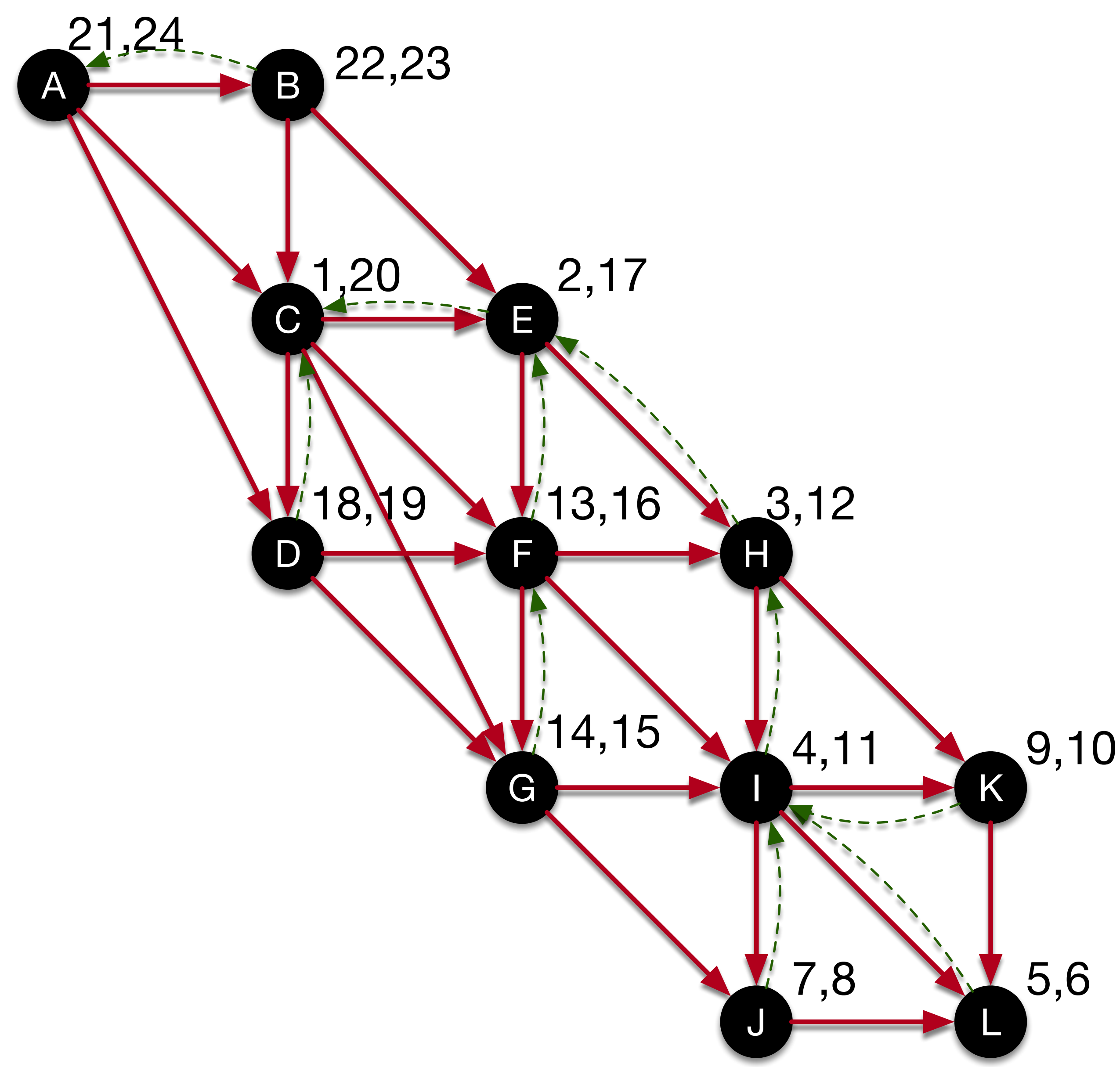



```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

We can finish B

Depth First Search

```
OS stack
   dfs_visit(A)
```

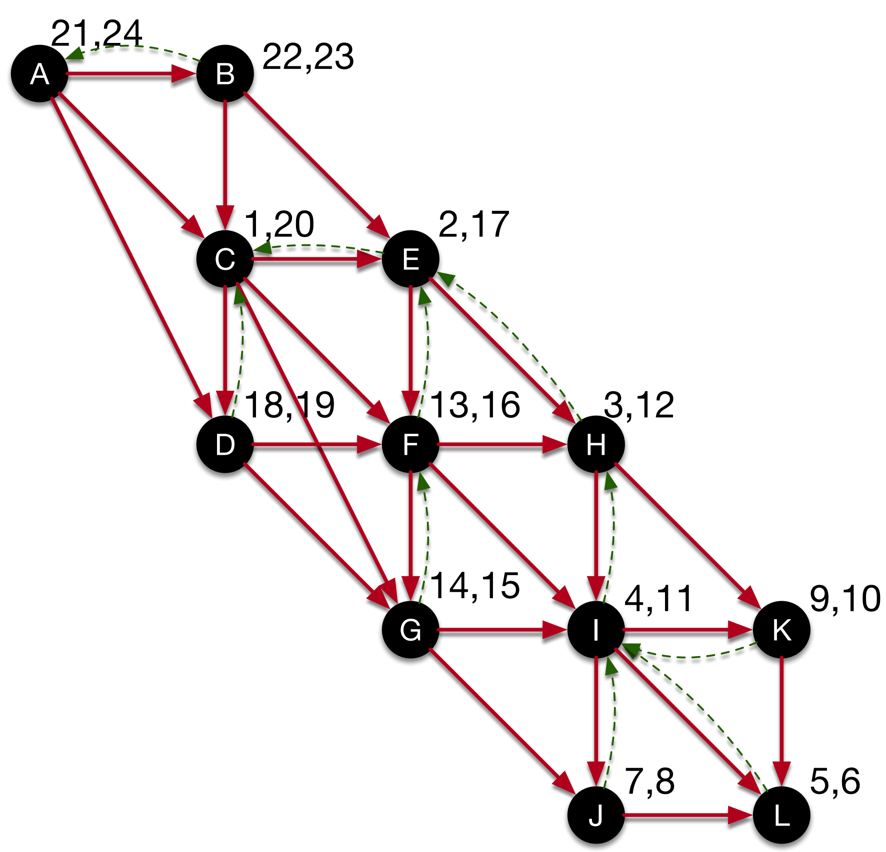


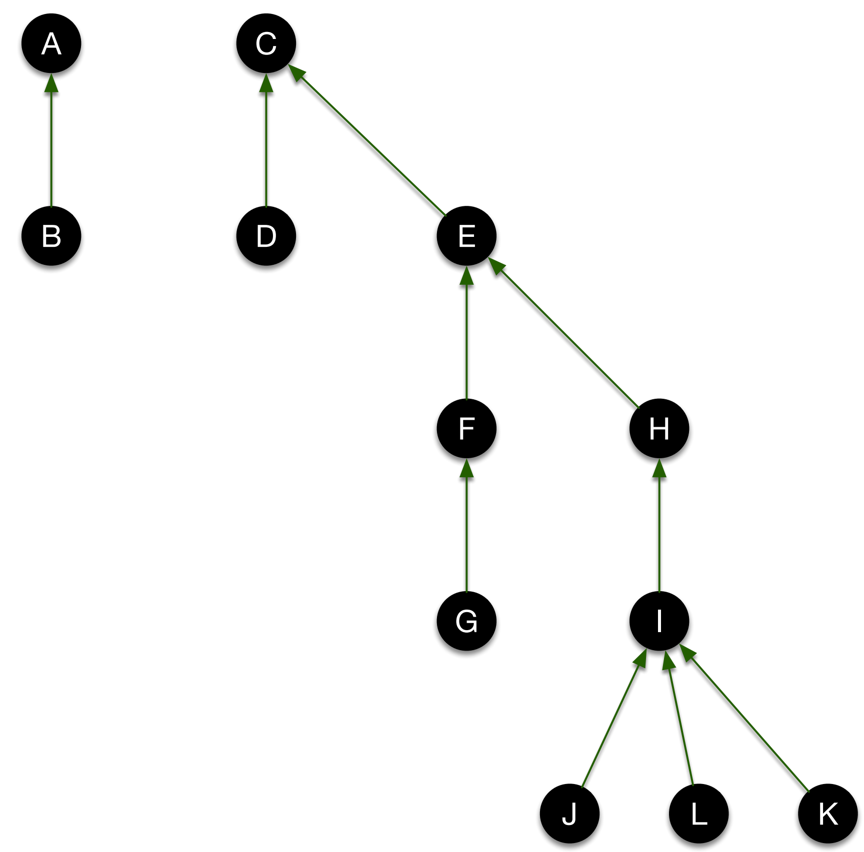

```
dfs_visit(u):
    u.color = 'gray'
    for each v in u.adjacency:
        if v.color == 'white'
            dfs_visit(v)
    u.color = 'black
```

We can finish A

Depth First Search

- Now we are done
  - The predecessor relationship has given us a nice set of trees — a "forest"





Depth First Search

- Runtime of algorithm
  - We look at all the elements of the adjacency lists
  - For each, we do constant work
  - But we also need to do some initial work for all vertices
  - Runtime is $\Theta(\max(|V|, |E|)$

Depth First Search

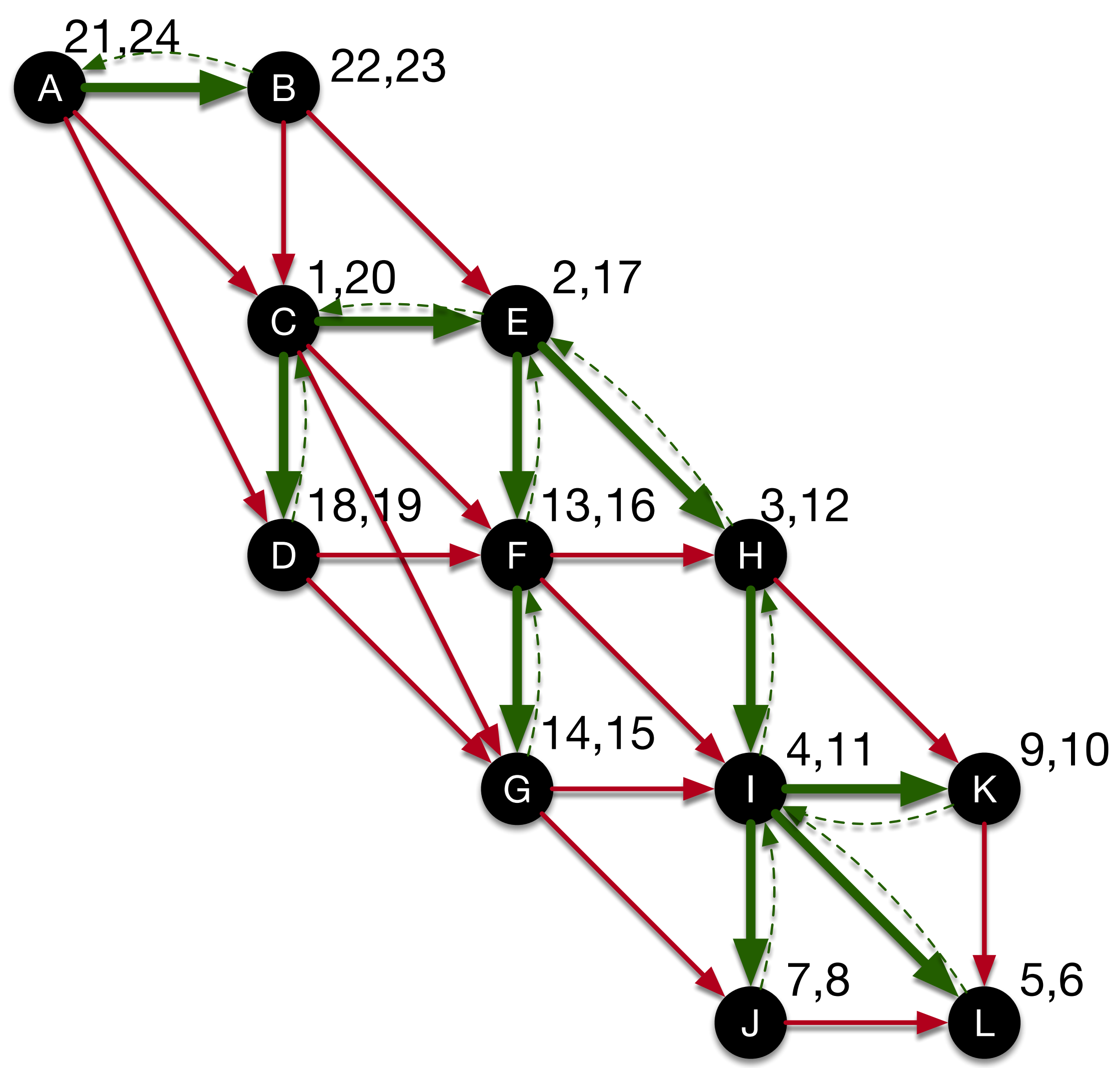
- Properties:
  - Parenthesis Theorem
    - If for two nodes
      - If $[u.d, u.f] \cap [v.d, v.f] = \emptyset$ then neither u and v are descendants in the predecessor forest
      - If $[u.d, u.f] \subset [v.d, v.f]$ then u is a descendant of v
      - If $[u.d, u.f] \supset [v.d, v.f]$ then v is a descendant of u

Depth First Search

- White Path Theorem
  - v is a descendant of u exactly if
  - At the time of discovery of u there is a path from u to v consisting entirely of white vertices

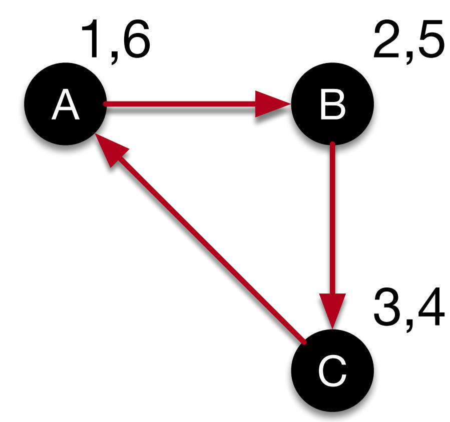
Depth First Search

- Classification of edges:
  - Tree edges are edges in the depth first tree



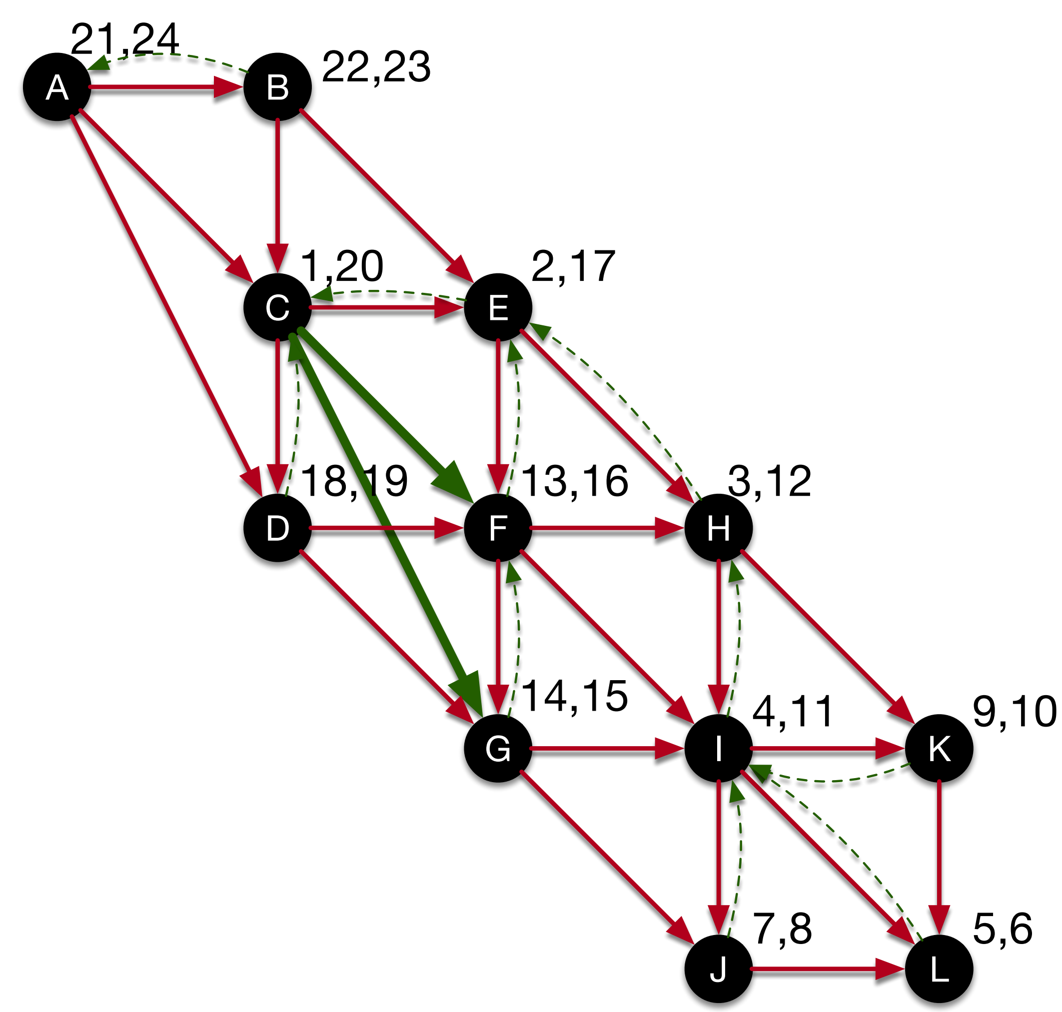
Depth First Search

- Back edges are edges go from a descendant to an ancestor
  - Simple example:
    - Start in A, discover B, discover C
    - Edge from C to A is a back edge



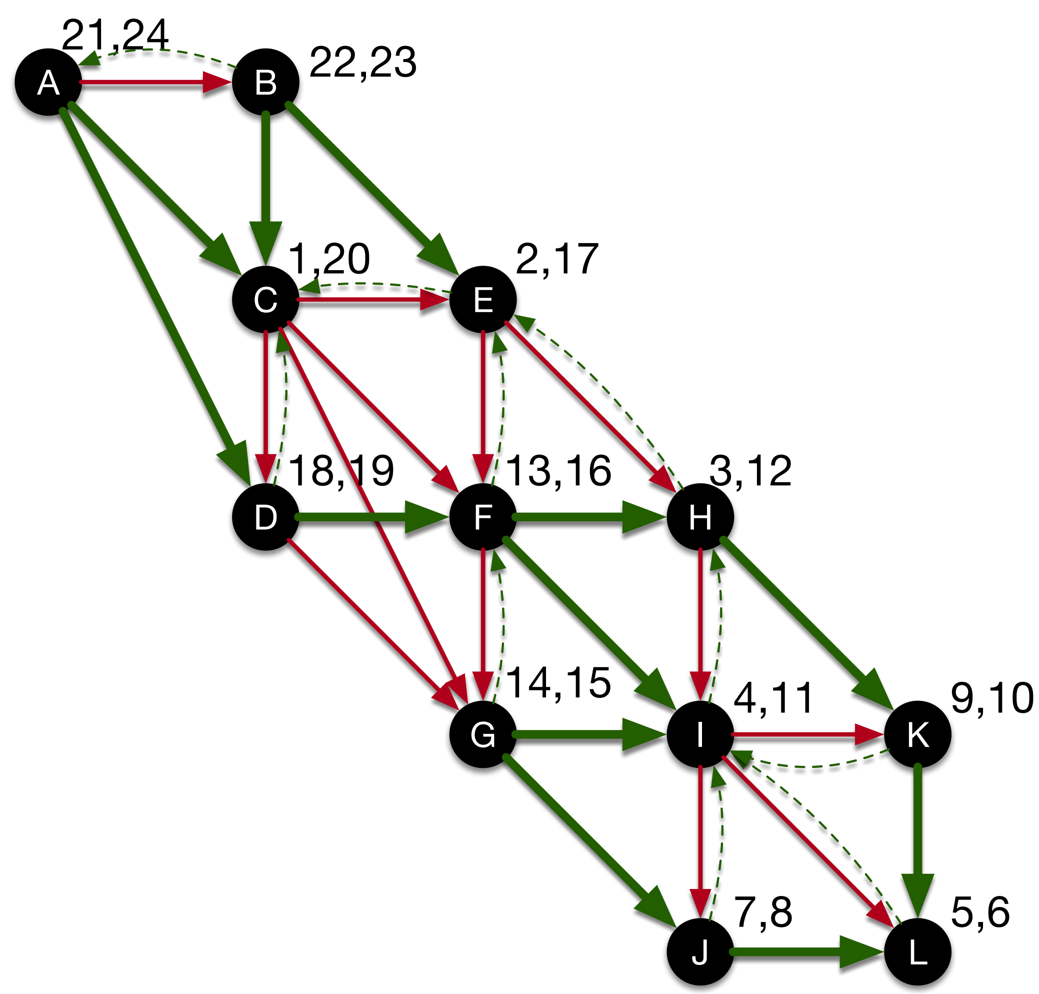
Depth First Search

- Forward edges
  - Edges connecting an ancestor to a descendant, but that are not in the tree



Depth First Search

- Cross Edges: anything else
  - Can be in the same tree or connecting different trees



Depth First Search

- If we look at an edge (u,v) during depth first search for the first time
  - (In an undirected graph, we look at each edge twice)
  - If v is white: tree edge
  - If v is gray: back edge
  - If v is black: forward or cross edge

Depth First Search

- In a depth first search on an undirected graph, every edge is either a tree edge or a back edge
  - Let (u, v) be an edge and assume that u is discovered first: $u.d < v.d$
  - The algorithm discovers and finishes v before u, so $u.f > v.f$
  - If DFS uses the edge (u, v) from u, then v is white, and (u, v) becomes a tree edge
  - If DFS uses the edge (u, v) from v, then u is gray at this moment and this becomes a back edge.