

Recursion

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“The power of recursion evidently lies in the possibility of defining an infinite set of objects by a finite statement. In the same manner, an infinite number of computations can be described by a finite recursive program, even if this program contains no explicit reference.”

ALGORITHMS + DATASTRUCTURES = PROGRAMS

N.E. WIRTH, 1976

Recursion as a Universal Tool

- Recursion possible:
 - Solution depends partially on solution(s) to (a) smaller problem(s)
- Recursion function consists of
 - Base Case
 - Call to function with smaller arguments

Examples for Recursion

- Euclidean Algorithm
 - Base case: one number is zero
 - Recursion: express the problem using smaller numbers
 - $a > b \Rightarrow \text{gcd}(a, b) = \text{gcd}(b, a \% b)$

Examples of Recursion:

Efficient Calculations of Powers

- Naive power calculation:

- ```
def power(x, n):
 acc = 1
 for _ in range(n):
 acc *= x
```

- This uses  $n$  multiplications.
- Can do better by setting `acc = x`, but that still uses  $n - 1$  multiplications

# Examples of Recursion:

## Efficient Calculations of Powers

- There is a better way with recursion
  - If  $n$  is even,  $n = 2m$ :  $x^n$  is the product of  $x^m$  with itself.
  - If  $n$  is odd,  $n = 2m + 1$ :  $x^n$  is  $x^m \cdot x^m \cdot x$
- This leads to very simple Python code

# Examples of Recursion:

## Efficient Calculations of Powers

- Direct Python Implementation:

```
def power(x, n):
 if n == 0:
 return 1
 if n == 1:
 return x
 if n%2 == 0:
 return power(x, n//2) * power(x, n//2)
 return power(x, n//2) * power(x, n//2) * x
```

# Examples of Recursion:

## Efficient Calculations of Powers

- Why does this work
  - A formal proof can assure that we did not make an implementation mistake
  - Proof is by induction
    - Base Cases:  $n=0$  and  $n=1$  are directly in the code
    - Induction Step: Assume it works for all inputs up to, but not including  $n$ 
      - Need to show that it also works for  $n$



# Examples of Recursion:

## Efficient Calculations of Powers

- Case distinction:
  - If  $n$  is even:
    - Then  $x^n = x^{n/2} \cdot x^{n/2}$  and the code works
  - If  $n$  is odd
    - Then  $x^n = x \cdot x^{n/2} \cdot x^{n/2}$  and the code works
- Here, we are using the induction hypothesis

# Examples of Recursion:

## Efficient Calculations of Powers

- As you can see, recursion and induction match each other closely

# Examples of Recursion:

## Efficient Calculations of Powers

- Performance:
  - Best case:  $n$  is a power of 2, i.e.  $n = 2^m$ 
    - By induction: show that the algorithm takes  $m$  steps.
    - Each step uses one multiplication.
    - Total of  $m = \log_2(n)$  multiplications

# Examples of Recursion:

## Efficient Calculations of Powers

- Performance:
  - Worse case:  $n$  is always odd
    - $n = 1, 3, 7, \dots = a_1, a_2, a_3, \dots$
    - Next element in this sequence is calculated from the previous:  $a_j = 2 \cdot a_{j-1} + 1$
  - Can prove by induction:
    - $a_j = 2^j - 1$

# Examples of Recursion:

## Efficient Calculations of Powers

- Performance:
  - Worst case:  $n = 2^m - 1$
  - Two multiplications per step
  - There are  $m - 1$  steps
  - Total is  $2m - 2$  multiplications
  - Which is  $2 \log_2(n) - 2$  multiplications

# Examples of Recursion:

## Efficient Calculations of Powers

- Performance:  $O(\log(n))$

# Examples of Recursion:

## Efficient Calculations of Powers

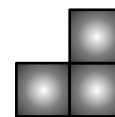
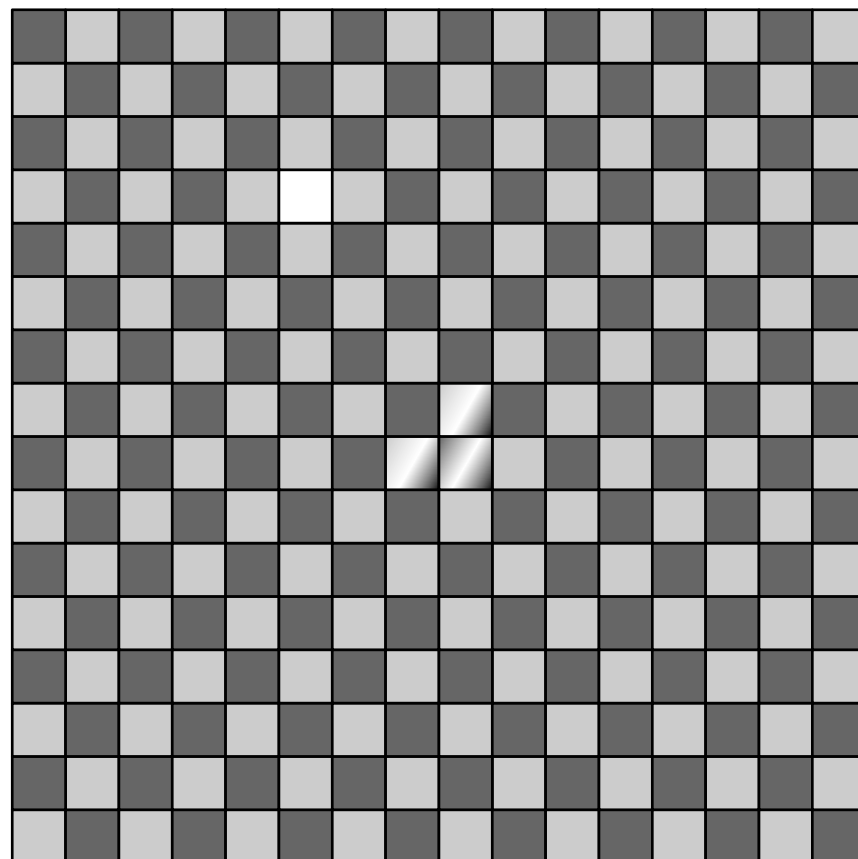
- Implementation using binary operations

```
def power(x, n):
 if n == 0:
 return 1
 if n == 1:
 return x
 m = n >> 1
 r = n & 0x01
 p = power(x, m)
 if r:
 return p*p*x
 return p*p
```

# Examples of Recursion:

## Triominos

- We are given a chess board of size  $2^m \times 2^m$  with one field removed.
- Write a program that shows how to tessellate the chess board with a triomino





# Examples of Recursion:

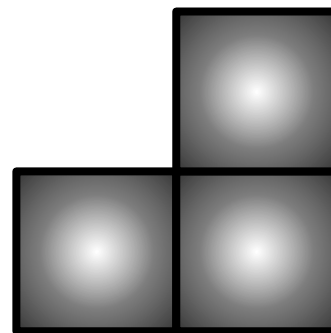
## Triominos

- Notice:
  - The chess board has  
 $2^m \times 2^m \equiv (-1)^m \times (-1)^m \equiv 1 \pmod{3}$  fields.
  - A Triomino has three fields
  - So, maybe it is possible

# Examples of Recursion:

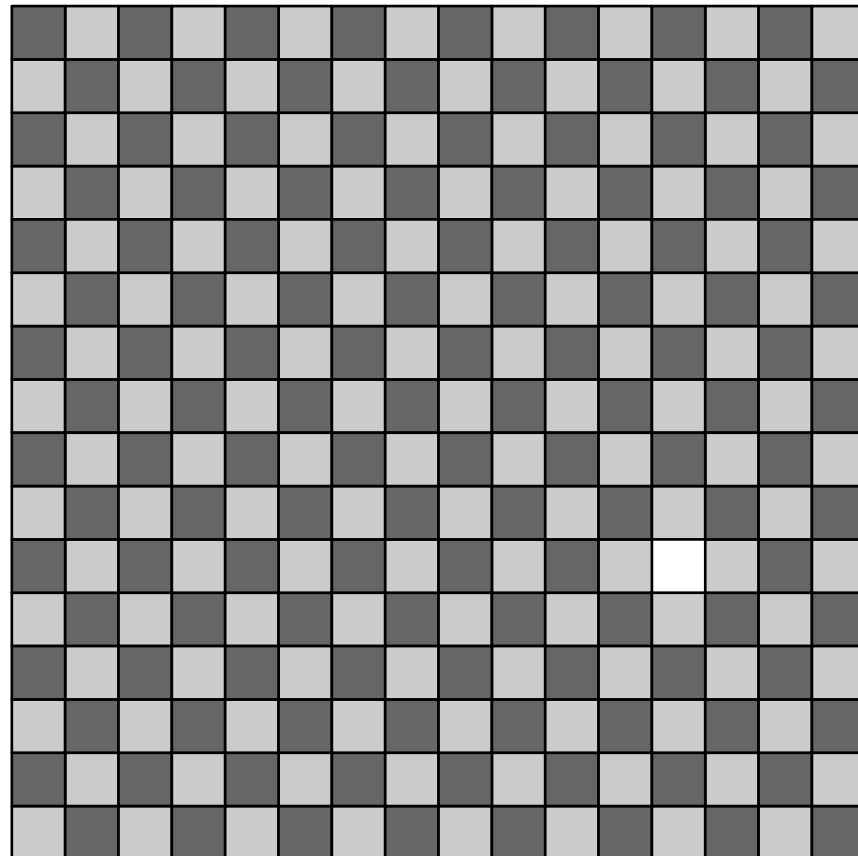
## Triominos

- Base Case:
  - $m = 1$ .
    - A two-by-two chess board has four fields.
    - Remove one, and you have a triomino



# Examples of Recursion: Triominos

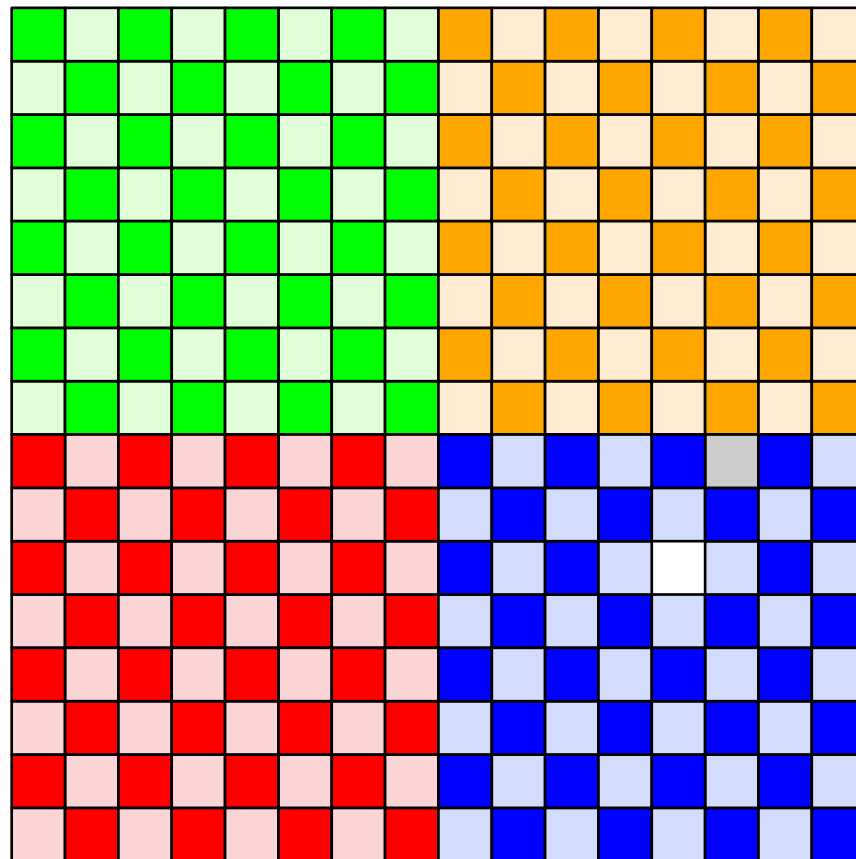
- Recursion:
  - A  $2^m \times 2^m$  chessboard consists of four  $2^{m-1} \times 2^{m-1}$  chessboards.
  - Take such a board with one field removed.



# Examples of Recursion:

## Triominos

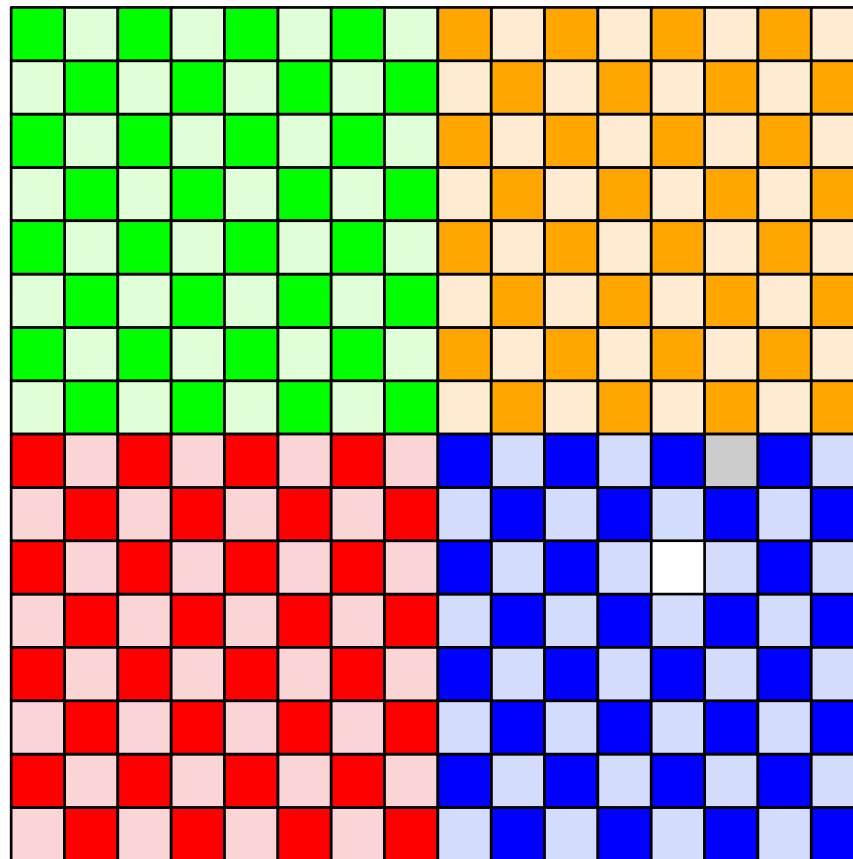
- Divide the board into four equal parts



# Examples of Recursion:

## Triominos

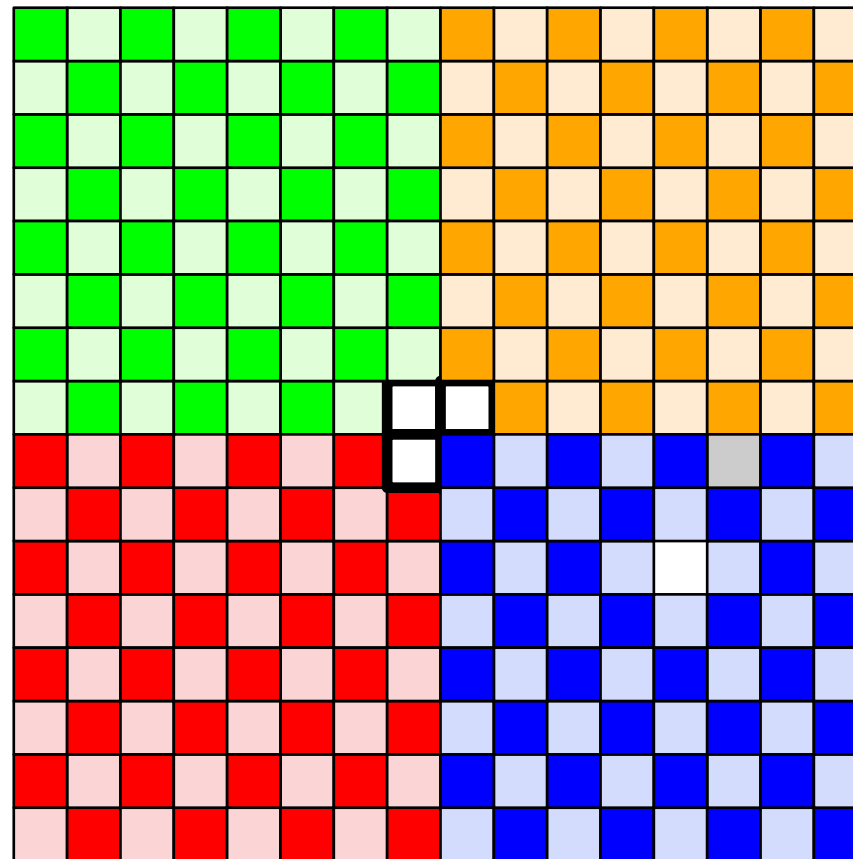
- One of them (here the blue one) has the missing field.
- Create a list of triominos that fill this up



# Examples of Recursion:

## Triominos

- Place a Triomino in the middle, cutting out one field from each of the other sub-boards.



# Examples of Recursion: Triominos

- Add this triomino to the list.
- The three remaining boards (green, orange, red) can now be covered with triominos

