Homework 1 Solution

Show that the recursion $x_1 = 0,\ x_2 = 1,\ x_{n+1} = 2x_n - x_{n-1}$ can only be solved by $x_n = n - 1$.

Proof by induction:

Base cases: $x_1 = 0$ by definition and $x_1 = 1 - 1 = 0$. Similarly $x_2 = 1$ by definition and $x_2 = 2 - 1 = 1$.

Induction Step:

Induction Hypothesis: $\forall m \in \mathbb{N}, m \leq n \ :\ x_m = m - 1$.

(Aside: we need two substitutions from the induction hypothesis, so we better use the stronger form of induction. This is also the reason why we need two base cases.)

Now, we have to show that for $n \geq 2$ we have $x_n = n - 1$. It is easier notation-wise to prove $x_{n+2} = (n+2) - 1$ with $n \geq 0$.

Proof:

$$\begin{aligned} x_{n+2} &= 2x_{n+1} - x_n && \text{Recursion equation} \\ &= 2(n+1-1) - (n-1) && \text{Induction Hpyothesis applied twice} \\ &= 2n - n + 1 \\ &= n + 1 \\ &= (n+2) - 1 \end{aligned}$$